THE UNIVERSITY OF ALBERTA

THE APPROXIMATE DISTRIBUTION OF THE REGRESSION COEFFICIENT BETWEEN TWO STATIONARY, LINEAR, MARKOV SERIES

by

© RICHARD ROUTLEDGE

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE

DEGREE OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

FALL, 1972

72F-195

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled "The Approximate Distribution of the Regression Coefficient Between Two Stationary, Linear, Markov Series" submitted by Richard Routledge in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

An approximate density function for the classical regression coefficient when the residuals and the "controlled variable" are independent, Gaussian Markov series is derived and is compared to the classical density function for the regression coefficient.

ACKNOWLEDGEMENTS

I should like to thank Dr. J. R. McGregor for suggesting the problem and for his considerate guidance in the work. I should further like to thank Mrs. Mary Willard for her assistance in the writing of the computer programs.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

1.1 Introduction and Summary

The simple autoregressive model,

$$y_i = \alpha_1 + \beta_1 x_i + e_i,$$

where $x_i = \rho_1 x_{i-1} + \varepsilon_i$,

and $e_i = \rho_2 e_{i-1} + \eta_i$,

and the ε's and η's are independent, standard normal random variables, frequently occurs in the analysis of time series. Several methods have been proposed for estimating the parameter, β_1 (see, e.g., Malinvaud (1966)). Suppose, however, that the classical, least squares procedure is used; that is, suppose that, in the case where $E(y_i)$ and $E(x_i)$ are known, the origin is shifted to make them 0, and β_1 is estimated by $b_1 = (\Sigma x_i y_i)/(\Sigma x_i^2)$[†], and in the case where the means are unknown, β_1 is estimated by

$$b_1^* = [\Sigma(x_i-\bar{x})(y_i-\bar{y})]/\Sigma(x_i-\bar{x})^2.$$

[†] All summations, unless explicitly stated otherwise, are from i equals 1 to n.

What properties do these classical estimators still have?

Watson (1951) has treated a similar problem, multiple linear regression with deterministic explanatory variables. He found that the classical estimators were unbiased and consistent. Unbiasedness is not hard to show in the model under discussion here.

First note that

$$b_1 = \frac{\Sigma x_i y_i}{\Sigma x_i^2} = \frac{\Sigma x_i(\beta_1 x_i + e_i)}{\Sigma x_i^2} = \beta_1 + \frac{\Sigma x_i e_i}{\Sigma x_i^2} \quad ,$$

and that

$$b_1^* = \frac{n\Sigma x_i y_i - (\Sigma x_i)(\Sigma y_i)}{n\Sigma x_i^2 - (\Sigma x_i)^2}$$

$$= \frac{n\Sigma x_i(\alpha_1 + \beta_1 x_i + e_i) - (\Sigma x_i)\Sigma(\alpha_1 + \beta_1 x_i + e_i)}{n\Sigma x_i^2 - (\Sigma x_i)^2}$$

$$= \beta_1 + \frac{n\Sigma x_i e_i - (\Sigma x_i)(\Sigma e_i)}{n\Sigma x_i^2 - (\Sigma x_i)^2} \quad .$$

Thus, the only contribution β_1 makes to the distributions of b_1 and b_1^* is a shift, and hence, b_1 and b_1^* are unbiased for β_1 if $(b_1-\beta_1)$ and $(b_1^*-\beta_1)$, both of which are independent of β_1, have zero expectation. Furthermore, since the last two statistics are used in testing hypotheses and setting up confidence intervals, we can restrict our attention to the

case where $\beta_1 = 0$. Moreover, both b_1 and b_1^* are independent of α_1, and henceforth we may assume, without loss of generality, that $\alpha_1 = 0$. With these restrictions, observe that

$$y_i = e_i \ ,$$

$$\begin{aligned} E(b_1) &= E\{E\{b_1 | x_1,\ldots,x_n\}\} \\ &= E\{E\{(\Sigma x_i y_i)/(\Sigma x_i^2) | x_1,\ldots,x_n\}\} \\ &= E\{(\Sigma x_i^2)^{-1} \times \Sigma x_i E(y_i)\} \\ &= 0 \ , \end{aligned}$$

and

$$\begin{aligned} E(b_1^*) &= E\{E\{b_1^* | x_1,\ldots,x_n\}\} \\ &= E\{E\{[n\Sigma x_i y_i - (\Sigma x_i)(\Sigma y_i)]/[\Sigma x_i^2 - (\Sigma x_i)^2] \\ &\qquad\qquad | x_1,\ldots,x_n\}\} \\ &= E\{[n\Sigma x_i^2 - (\Sigma x_i)^2]^{-1} \times [n\Sigma x_i E(y_i) - (\Sigma x_i)(\Sigma E(y_i))]\} \\ &= 0 \ . \end{aligned}$$

Hence, b_1 and b_1^* are unbiased. Furthermore, using the fact that the x's are independent of the y's, and that the x's are symmetrically distributed about zero, intuitively, one would expect b_1 and b_1^* to be symmetrically distributed about zero.

In the next section of this chapter, the distributions of b_1 and b_1^* are derived in the classical case of independent observations. Chapter II deals with the more general case of non-zero serial correlations. McGregor's (1962) derivation of the approximate joint density function for the two regression coefficients for the "known means" case, and Bielenstein's (1963) corresponding result for the "fitted means" case are derived in the first four sections of Chapter II. In the fifth section, these joint densities are integrated to obtain the approximate densities of b_1 and b_1^*. It is then shown that these densities reduce to the classical distributions of section 1.2 when $\rho_1 = \rho_2 = 0$. In the sixth section, approximate variances are calculated.

The results of a numerical renormalization with some sample graphs are presented in Chapter III, along with the results of a simulation to test the accuracy of the approximation.

Programs used in the numerical renormalization and simulation are presented in the appendix.

1.2 The Classical Densities of the Regression Coefficients

Suppose a simple random sample of n observations is chosen from a bivariate normal population with mean, $\underline{\mu} = (\mu_1, \mu_2)$,

and covariance matrix, $\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$.

If the mean vector is known, then it is subtracted from all the observations. Denote the resulting sample by

$$(x_1, y_1), \ldots, (x_n, y_n).$$

Then, the joint distribution of the sample is

$$dF_{x_1, y_1, \ldots, x_n, y_n} = [2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}]^{-n}$$

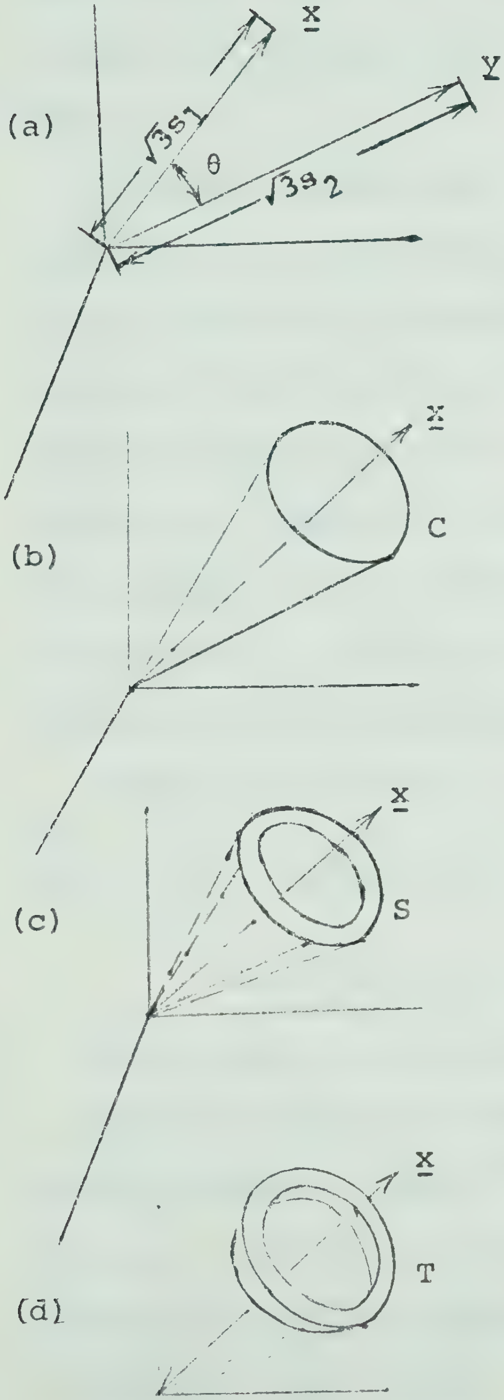
$$\times \exp\left[\frac{-1}{2(1-\rho^2)}\left[\frac{\Sigma x_i^2}{\sigma_1^2} - \frac{2\rho\Sigma x_i y_i}{\sigma_1\sigma_2} + \frac{\Sigma y_i^2}{\sigma_2^2}\right]\right] dx_1 \ldots dy_n$$

$$= [2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}]^{-n}$$

$$\times \exp\left[\frac{-n}{2(1-\rho^2)}\left[\frac{s_1^2}{\sigma_1^2} - \frac{2\rho r s_1 s_2}{\sigma_1\sigma_2} + \frac{s_2^2}{\sigma_2^2}\right]\right] dx_1 \ldots dy_n \ , \quad (1.2.1)$$

where $s_1^2 = (\Sigma x_i^2)/n$, $s_2^2 = (\Sigma y_i^2)/n$, and $r = (\Sigma x_i y_i)/n s_1 s_2$.

To obtain the joint distribution of the statistics, s_1, s_2, and r, we need to transform the differential volume element, $dx_1 \ldots dy_n$, to an expression depending on s_1, s_2, r, ds_1, ds_2, and dr. A geometric approach, first used by Fisher (1915) to obtain a corresponding result for the case where $\underline{\mu}$ is unknown, is most appropriate.



For fixed $\underline{x}$, r, and s_2, $\underline{y}$ is constrained to lie on the circle, C, of radius $\sqrt{3}s_2 \sin\theta$.

If r is allowed to vary to r + dr $\underline{y}$ can lie anywhere on the annular section, A, of the surface of the sphere with centre O and radius $\sqrt{3}s_2$.

If, furthermore, s_2 is allowed to vary to $s_2 + ds_2$, $\underline{y}$ can lie anywhere in the toroidal region, T.

Figure 1: The Differential Volume Element for n = 3.

Since $r = (\Sigma x_i y_i)/[(\Sigma x_i^2)(\Sigma y_i^2)]^{1/2}$, r can be considered as the cosine of the angle, say θ, between $\underline{x}$ and $\underline{y}$. Furthermore, $\sqrt{n}s_1 = [\Sigma x_i^2]^{1/2}$ = the length of $\underline{x}$, and $\sqrt{n}s_2$ = the length of $\underline{y}$. (See Figure 1(a), for the three-dimensional case.)

If $\underline{x}$ and r are fixed, then $\underline{y}$ can vary over the cone of angle θ to $\underline{x}$. (See Figure 1(b).) If s_2 is also fixed, then $\underline{y}$ is constrained to range over an (n-2)-dimensional spherical surface of radius $\sqrt{n}s_2\sin(\theta) = \sqrt{n}s_2(1-r^2)^{1/2}$. If the correlation is allowed to vary from r to r+dr, then $\underline{y}$ can lie anywhere on a spherical shell of thickness $\sqrt{n}s_2 d = \sqrt{n}s_2(1-r^2)^{1/2}dr$. (See Figure 1(c).) Finally, if the length of $\underline{y}$ is allowed to vary from $\sqrt{n}s_2$ to $\sqrt{n}(s_2+ds_2)$, then the set of allowable values of $\underline{y}$ has a further dimension, say a width, of size $\propto ds_1$

Hence, for fixed $\underline{x}$,

$$dy_1 \ldots dy_n \propto [s_2^2(1-r^2)]^{(n-2)/2} s_2(1-r^2)^{-1/2} dr ds_2$$

$$= s_2^{n-1}(1-r^2)^{(n-3)/2} dr ds_2 \ .$$

For fixed s_1, $\underline{x}$ is constrained to an (n-1)-dimensional spherical surface of radius $\sqrt{n}s_1$, and hence of (n-1)-dimensional volume $\propto (\sqrt{n}s_1)^{n-1} \propto s_1^{n-1}$. If the length of $\underline{x}$ is allowed to vary from $\sqrt{n}s_1$ to $\sqrt{n}(s_1+ds_1)$, $\underline{x}$ can be anywhere on a spherical shell of approximately the same radius and of thickness $\sqrt{n}ds_1$. Hence, $dx_1 \ldots dx_n \propto s_1^{n-1}ds_1$ and

$$dx_1 \ldots dx_n dy_1 \ldots dy_n \propto s_1^{n-1} s_2^{n-2} (1-r^2)^{(n-3)/2} ds_1 ds_2 dr. \quad (1.2.2)$$

Now, using (1.2.1) and (1.2.2), the joint distribution

of s_1, s_2, and r is given by

$$dF_{s_1 s_2 r} \propto \exp\left[\frac{-n}{2(1-\rho^2)}\left[\frac{s_1^2}{\sigma_1^2} - \frac{2\rho r s_1 s_2}{\sigma_1\sigma_2} + \frac{s_2^2}{\sigma_2^2}\right]\right]$$

$$\times\ s_1^{n-1} s_2^{n-2}(1-r^2)^{(n-3)/2} ds_1 ds_2 dr \ . \qquad (1.2.3)$$

Transforming the variables s_1, s_2, and r to s_1, s_2, and

$b_1 = rs_2/s_1$, with Jacobian,

$$\frac{\partial(s_1,s_2,r)}{\partial(s_1,s_2,b_1)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \dfrac{b_1}{s_2} & \dfrac{-b_1 s_1}{s_2^2} & \dfrac{s_1}{s_2} \end{vmatrix} = \frac{s_1}{s_2} \ ,$$

the joint distribution of s_1, s_2, and b_1 is found to be

$$dF_{s_1 s_2 b_1} \propto \exp\left[\frac{-n}{2(1-\rho^2)}\left[\frac{s_1^2}{\sigma_1^2} - \frac{2\rho b_1 s_1^2}{\sigma_1\sigma_2} + \frac{s_2^2}{\sigma_2^2}\right]\right]$$

$$\times\ s_1^{n} s_2^{n-2}(1-b_1 s_1/s_2)^{(n-3)/2}\ ds_1 ds_2 db_1 \ .$$

Noting that $r^2 \leqq 1$, and hence, $b_1^2 s_1^2/s_2^2 \leqq 1$,

and $s_2^s \geqq s_1^2 b_1^2$, the joint distribution of b_1 and s_1 is found

to be

$$dF_{s_1,b_1} \propto \int_{s_1|b_1|}^{\infty} \exp\left[\frac{-n}{2(1-\rho^2)}\left[\frac{s_1^2}{\sigma_1^2} - \frac{2\rho b_1 s_1^2}{\sigma_1\sigma_2} + \frac{s_2^2}{\sigma_2^2}\right]\right]$$

$$\times\ s_1^{n} s_2^{n-2}(1-b_1^2 s_1^2/s_2^2)^{(n-3)/2}\ ds_1 ds_2 db_1$$

$$= ds_1 db_1 s_1^n \exp\left[\frac{-n}{2(1-\rho^2)}\left[\frac{s_1^2}{\sigma_1^2} - \frac{2\rho b_1 s_1^2}{\sigma_1\sigma_2}\right]\right]$$

$$\times \int_{s_1|b_1|}^{\infty} \exp\left[\frac{-ns_2^2}{2(1-\rho^2)\sigma_2^2}\right] s_2^{n-2}(1-b_1^2s_1^2/s_2^2)^{(n-3)/2} ds_2. \quad (1.2.4)$$

Substituting $u = s_2^2 - s_1^2 b_1^2$, with $du = 2s_2 ds_2$, the integral in (1.2.4) becomes

$$\frac{1}{2}\int_0^{\infty} \exp\left[\frac{-n(u+b_1^2s_1^2)}{2(1-\rho^2)\sigma_2^2}\right] s_2^{n-3}\left[\frac{1}{s_2^2}\right]^{(n-3)/2} \times u^{(n-3)/2}\, du$$

$$= \frac{1}{2}\exp\left[\frac{-nb_1^2s_1^2}{2(1-\rho^2)\sigma_2^2}\right]\int_0^{\infty} \exp\left[\frac{-nu}{2(1-\rho^2)\sigma_2^2}\right] u^{(n-3)/2}\, du\ . \quad (1.2.5)$$

Substituting $v = \dfrac{nu}{2(1-\rho^2)\sigma_2^2}$, with $dv = \dfrac{ndu}{2(1-\rho^2)\sigma_2^2}$,

the integral in (1.2.5) becomes

$$[2(1-\rho^2)\sigma_2^2/n]^{(n-1)/2} \int_0^{\infty} e^{-v}\, v^{(n-3)/2}\, dv$$

$$= [2(1-\rho^2)\sigma_2^2/n]^{(n-1)/2}\ \Gamma[(n-1)/2]\quad . \quad (1.2.6)$$

Hence, from (1.2.4), (1.2.5), and (1.2.6),

$$dF_{s_1,b_1} \propto \exp\left[\frac{-n}{2(1-\rho^2)}\left[\frac{s_1^2}{\sigma_1^2} - \frac{2\rho b_1 s_1^2}{\sigma_1\sigma_2} + \frac{b_1^2 s_1^2}{\sigma_2^2}\right]\right] s_1^n\, ds_1 db_1. \quad (1.2.7)$$

Finally, integrating (1.2.7) over the range of s_1, the distribution of b_1 is obtained as

$$dF_{b_1} \propto \int_0^\infty \exp\left[\frac{-n}{2(1-\rho^2)}\left[\frac{s_1^2}{\sigma_1^2} - \frac{2\rho b_1 s_1^2}{\sigma_1\sigma_2} + \frac{b_1^2 s_1^2}{\sigma_2^2}\right]\right] s_1^n \, ds_1 db_1$$

$$= db_1 \int_0^\infty \exp\left[\frac{-ns_1^2}{2(1-\rho^2)}\left[\frac{1}{\sigma_1^2} - \frac{2\rho b_1}{\sigma_1\sigma_2} + \frac{b_1^2}{\sigma_2^2}\right]\right] s_1^n \, ds_1$$

$$= \frac{db_1}{2}\left[\frac{2(1-\rho^2)}{n\left[\frac{1}{\sigma_1^2} - \frac{2\rho b_1}{\sigma_1\sigma_2} + \frac{b_1^2}{\sigma_2^2}\right]}\right]^{(n+1)/2} \int_0^\infty e^{-t}\, t^{(n-1)/2}\, dt$$

$$= \frac{1}{2}\cdot\left[\frac{2(1-\rho^2)\sigma_1^2\sigma_2^2}{n[\sigma_2^2 - 2\rho b_1\sigma_1\sigma_2 + b_1^2\sigma_1^2]}\right]^{(n+1)/2} \Gamma[(n+1)/2]\, db_1 \ ,$$

where $t = \dfrac{ns_1^2}{2(1-\rho^2)}\left[\dfrac{1}{\sigma_1^2} - \dfrac{2\rho b_1}{\sigma_1\sigma_2} + \dfrac{b_1^2}{\sigma_2^2}\right]$.

That is, $$dF_{b_1} \propto \frac{db_1}{[\sigma_2^2-2\rho b_1\sigma_1\sigma_2+b_1^2\sigma_1^2]^{(n+1)/2}} \quad . \qquad (1.2.8)$$

To evaluate the constant, first make the substitution,

$u = b_1 - \beta_1 = b_1 - \dfrac{\rho\sigma_2}{\sigma_1}$, with $du = db_1$.

The denominator of (1.2.8) becomes

$$\begin{aligned} &[\sigma_2^2 - 2\rho(u+\rho\sigma_2/\sigma_1)\sigma_1\sigma_2 + (u+\rho\sigma_2/\sigma_1)^2\sigma_1^2]^{(n+1)/2} \\ &= [\sigma_2^2 - 2\rho\sigma_1\sigma_2 u - 2\rho^2\sigma_2^2 + u^2\sigma_1^2 + 2\rho u\sigma_1\sigma_2 + \rho^2\sigma_2^2]^{(n+1)/2} \\ &= [\sigma_2^2(1-\rho^2) + u^2\sigma_1^2]^{(n+1)/2} \\ &= [\sigma_2^2(1-\rho^2)]^{(n+1)/2}\ [1+u^2\sigma_1^2/(1-\rho^2)\sigma_1^2]^{(n+1)/2} \quad , \end{aligned}$$

and $$\int_{-\infty}^{\infty} [\sigma_2^2-2\rho b_1\sigma_1\sigma_2+b_1^2\sigma_1^2]^{-(n+1)/2}\, db_1$$

$$= [\sigma_2^2(1-\rho^2)]^{-(n+1)/2} \int_{-\infty}^{\infty} [1+u^2\sigma_1^2/(1-\rho^2)\sigma_2^2]^{-(n+1)/2}\, du \ . \qquad (1.2.9)$$

Making the final substitution,

$$v = \sigma_1\sqrt{n}u/\sigma_2(1-\rho^2)^{1/2} \ ,$$

the integral becomes

$$[\sigma_2(1-\rho^2)^{1/2}/\sigma_1\sqrt{n}]\int_{-\infty}^{\infty} (1+v^2/n)^{(n+1)/2}\, dv \ , \qquad (1.2.10)$$

which may be evaluated by complex integration or by

noting that the integrand is proportional to the t-density, with n degrees of freedom. Hence, (see, e. g., Kendall and Stuart (1958), equation (11.44))

$$\int_{-\infty}^{\infty}(1+v^2/n)^{(n+1)/2}\,dv = \sqrt{n\pi}\,\Gamma(n/2)/\Gamma[(n+1)/2],$$

and from (1.2.9) and (1.2.10), the constant is

$$[\sigma_2^2(1-\rho^2)]^{(n+1)/2} \times [(1-\rho^2)^{1/2}\sigma_2/\sqrt{n}\sigma_1]^{-1} \times \Gamma[(n+1)/2]/\sqrt{n\pi}\,\Gamma(n/2)$$

$$= \sigma_1\sigma_2^n(1-\rho^2)^{n/2}\Gamma[(n+1)/2]/\sqrt{\pi}\,\Gamma(n/2).$$

Hence, using (1.2.8), and simplifying,

$$dF_{b_1} = \frac{\sigma_1\sigma_2^n(1-\rho^2)^{n/2}\Gamma[(n+1)/2]}{\sqrt{\pi}\,\Gamma(n/2)}[\sigma_2^2 - 2\rho b_1\sigma_1\sigma_2 + b_1^2\sigma_1^2]^{-(n+1)/2}\,db_1$$

$$= \frac{\sigma_2^n(1-\rho^2)^{n/2}\Gamma[(n+1)/2]}{\sigma_1^n\sqrt{\pi}\,\Gamma(n/2)}[(\sigma_2^2/\sigma_1^2)(1-\rho^2) + (b_1 - \beta_1)^2]^{-(n+1)/2}$$

$$\times\, db_1. \qquad (1.2.11)$$

If $\rho = 0$, and $\sigma_1=\sigma_2= 1$, then (1.2.11) becomes

$$dF_{b_1} = \frac{\Gamma[(n+1)/2]}{\sqrt{\pi}\;\Gamma(n/2)}(1+b_1^2)^{-(n+1)/2}\,db_1\,. \qquad (1.2.12)$$

If the means are unknown, then the joint distribution of the sample is

$$dF_{x_1,\ldots,y_n} = [(2\pi)^n\sigma_1^n\sigma_2^n(1-\rho^2)^{n/2}]^{-1}\exp\{[-2(1-\rho^2)]^{-1}$$

(continued)

$$\times[\Sigma(x_i-\mu_1)^2/\sigma_1-2\rho[\Sigma(x_i-\mu_1)(y_i-\mu_2)]/\sigma_1\sigma_2+\Sigma(y_i-\mu_2)^2/\sigma_2^2]\}$$

$$\times dx_1\ldots dy_n$$

$$=[(2\pi)^n\sigma_1^n\sigma_2^n(1-\rho^2)^{n/2}]^{-1}\exp\{[-n/2(1-\rho^2)]\times[(\bar{x}-\mu_1)^2/\sigma_1^2$$

$$-2\rho(\bar{x}-\mu_1)(\bar{y}-\mu_2)/\sigma_1\sigma_2+(\bar{y}-\mu_2)^2/\sigma_2^2+s_1^2/\sigma_1^2 2\rho rs_1s_2/\sigma_1\sigma_2+s_2^2/\sigma_2^2]\}$$

$$\times dx_1\ldots dy_n, \qquad (1.2.13)$$

where $\bar{x}=\Sigma x_i/n$, $\bar{y}=\Sigma y_i/n$, $s_1^2=\Sigma(x_i-\bar{x})^2/n$, $s_2^2=\Sigma(y_i-\bar{y})^2/n$,

and $r=\Sigma(x_i-\mu_1)(y_i-\mu_2)/ns_1s_2$.

By a geometric argument similar to the one used in the 'known means' case, the joint density of $\bar{x}$, $\bar{y}$, s_1, s_2, and r can be obtained from (1.2.13). (For details of the derivation see Kendall and Stuart (1958), §§16.24-16.36.) This density can be factored into parts depending on $\bar{x}$ and $\bar{y}$, and s_1, s_2, and r respectively. Hence, s_1, s_2, and r are independent of $\bar{x}$ and $\bar{y}$. The joint distribution of s_1, s_2, and r is thereby found to be given by

$$dF_{s_1,s_2,r}\propto\exp\{-n/2(1-\rho^2)[s_1^2/\sigma_1^2-2\rho rs_1s_2/\sigma_1\sigma_2+s_2^2/\sigma_2^2]\}$$
$$\times s_1^{n-2}s_2^{n-2}(1-r^2)^{(n-4)/2}ds_1ds_2dr. \qquad (1.2.14)$$

Comparing this relation to (1.2.3), it is seen that they are identical except for a replacement of n by (n-1) in the last three terms before the differential element. Using similar substitutions, the distribution of b_1^*, the regression coef-

ficient with fitted means is found to be

$$dF_{b_1^*} = [\sigma_2^{n-1}(1-\rho^2)^{(n-1)/2}\ \Gamma(n/2)]\times[\sigma_1^{n-1}\sqrt{\pi}\Gamma[(n-1)/2]]^{-1}$$
$$\times\ [(\sigma_2^2/\sigma_1^2)(1-\rho^2) + (b_1^*-\beta_1)^2]^{-n/2}\ db_1^*\ . \qquad (1.2.15)$$

If $\rho = 0$, and $\sigma_1 = \sigma_2 = 1$, then (1.2.15) becomes

$$dF_{b_1^*} = \frac{\Gamma(n/2)}{\sqrt{\pi}\Gamma[(n-1)/2]}\ [1+b_1^{*2}]^{-n/2} \quad . \qquad (1.2.16)$$

CHAPTER II

APPROXIMATE DENSITIES FOR THE REGRESSION COEFFICIENTS

2.1 The Form of Two Relevant Generating Functions

To find the joint densities for the regression coefficients, and hence the marginal densities, the joint density of the x's and y's is first calculated. From this result, approximate expressions for the joint moment generating functions of Σx_i^2, $\Sigma x_i y_i$, and Σy_i^2 for the 'known means' case, and of $\Sigma(x_i-\bar{x})^2$, $\Sigma(x_i-\bar{x})(y_i-\bar{y})$, and $\Sigma(y_i-\bar{y})^2$ for the 'fitted means' case are derived. After deriving, in section 2, approximate expressions for the determinants involved in the generating functions, a general method, due to McGregor (1962), for obtaining the joint densities of the regression coefficients is presented in section 3. The method is carried out in section 4. In sections 5 and 6, approximate expressions for the marginal densities and their variances are derived.

With $\alpha_1=\beta_1=0$, $x_i = \rho_1 x_{i-1} + \varepsilon_i$,

and $\quad y_i = \rho_2 y_{i-1} + \eta_i$,

where the ε's and η's are independent normal (0,1).
The joint distribution of $\underline{x}' = (x_1,\ldots,x_n)$ is then

$$dF_1(x_1,\ldots,x_n) = (2\pi)^{-n/2}|\underline{\Sigma}_x|^{-1/2}\exp[(-1/2)\underline{x}'\underline{\Sigma}_x^{-1}\underline{x}\ dx_1,\ldots,dx_n, \qquad (2.1.1)$$

where $\underline{\Sigma}_x$ is the covariance matrix of $\underline{x}$.
Using induction, one can easily show that

$\mathrm{Var}(x_i) = (1-\rho_1^2)^{-1}$, for $i = 1,\ldots,n$,

and $\mathrm{Cov}(x_{1-k}, x_i) = (1-\rho_1^2)^{-1}\rho_1^{|k|}$ for $k = i-n,\ldots,i-1$,

and $i = 1,\ldots,n$.

Hence, the covariance matrix of $\underline{x}$ is

$$\underline{\Sigma}_x = (1-\rho_1^2)^{-1}\begin{bmatrix} 1 & \rho_1 & \cdot & \cdot & \cdot & \rho_1^{n-1} \\ \rho_1 & 1 & \cdot & \cdot & \cdot & \rho_1^{n-2} \\ \cdot & \cdot & \cdot & & & \vdots \\ \cdot & \cdot & & \cdot & & \vdots \\ \rho_1^{n-1} & \rho_1^{n-2} & \cdot & \cdot & \cdot & 1 \end{bmatrix} . \quad (2.1.2)$$

To find $\underline{\Sigma}_x^{-1}$, first define $\underline{I}_x$ by

$$\underline{I}_x = \begin{bmatrix} 1 & -\rho_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & -\rho_1 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \cdot & & & \vdots \\ & & & \cdot & \cdot & & \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix} .$$

Then, $\underline{\tilde{\Sigma}}_x = \underline{I}_x\underline{\Sigma}_x$ is the triangular matrix,

$$\begin{bmatrix} 1 & & & & & & \\ \rho_1 & 1 & & & (0) & & \\ \rho_1^2 & \rho_1^1 & & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & & \cdot & & & \\ \rho_1^{n-2} & \rho_1^{n-3} & \cdot & \cdot & \cdot & 1 & \\ \dfrac{\rho_1^{n-1}}{1-\rho_1^2} & \dfrac{\rho_1^{n-2}}{1-\rho_1^2} & \cdot & \cdot & \cdot & \dfrac{\rho_1}{1-\rho_1^2} & \dfrac{1}{1-\rho_1^2} \end{bmatrix} ,$$

whose determinant is $(1-\rho_1^2)^{-1}$.

Hence,

$$\tilde{\Sigma}_x^{-1} = \begin{bmatrix} 1 & & & & \\ -\rho_1 & 1 & & (0) & \\ 0 & -\rho_1 & & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & \cdot & \\ 0 & 0 & \cdot & \cdots & 1-\rho_1^2 \end{bmatrix},$$

and

$$\Sigma_x^{-1} = \tilde{\Sigma}_x^{-1} I_x = \begin{bmatrix} 1 & -\rho_1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -\rho_1 & 1+\rho_1^2 & -\rho_1 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & -\rho_1 & 1+\rho_1^2 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1-\rho_1^2 & -\rho_1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -\rho_1 & 1 \end{bmatrix}. \tag{2.1.3}$$

Furthermore, $|\Sigma_x| = |\tilde{\Sigma}_x||I_x| = (1-\rho_1^2)^{-1}$.

Similarly, the joint distribution of $\underline{y}' = (y_1,\ldots,y_n)$ is

$$dF_2(y_1,\ldots,y_n) = (2\pi)^{-n/2}|\Sigma_y|^{-1/2}\exp\{(-1/2)\underline{y}'\Sigma_y^{-1}\underline{y}\ dy_1,\ldots,dy_n, \tag{2.1.4}$$

where Σ_y is the covariance matrix of $\underline{y}$, whose inverse is identical to Σ_x^{-1}, with ρ_1 replaced everywhere by ρ_2. (2.1.5)

Since $\underline{x}$ and $\underline{y}$ are independent, the distribution of $\underline{x}^{*\prime} = (\underline{x}'|\underline{y}')$ is

$$dF(x_1,\ldots,x_n,y_1,\ldots,y_n) = (1-\rho_1^2)^{1/2}(1-\rho_1^2)^{1/2}(2\pi)^{-n}$$

$$\times \exp\{(-1/2)\underline{x}^{*\prime}\hat{\underline{A}}\underline{x}^*\}dx_1,\ldots,dy_n,$$

$$\text{where } \underline{\hat{A}} = \left[\begin{array}{c|c} \underline{\Sigma}_x^{-1} & 0 \\ \hline 0 & \underline{\Sigma}_y^{-1} \end{array}\right] . \qquad (2.1.6)$$

$$\text{Define } C = \Sigma x_i^2,\ D = \Sigma y_i^2,\ E = \Sigma x_i y_i, \qquad (2.1.7)$$

$$\text{and } C^* = \Sigma(x_i-\bar{x})^2,\ D^* = \Sigma(y_i-\bar{y})^2, \text{ and } E^* = \Sigma(x_i-\bar{x})(y_i-\bar{y}). \quad (2.1.8)$$

Then the joint moment generating function of C, D, and E is

$$\begin{aligned} M(T,S,U) &= E[\exp(TC+SD+UE)] \\ &= (1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}(2\pi)^{-n} \\ &\quad \times\int\ldots\int \exp[TC+SD+UE-(1/2)\underline{x}^{*\prime}\underline{\hat{A}}\underline{x}^*]dx_1\ldots dy_n . \end{aligned} \qquad (2.1.9)$$

But $-2(TC+SD+UE)$

$$=-2[\Sigma T x_i^2+\Sigma S y_i^2+\Sigma U x_i y_i]$$

$$= \underline{x}^{*\prime}\left[\begin{array}{cccccc|cccccc} -2T & 0 & \cdot & \cdot & \cdot & 0 & -U & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & -2T & \cdot & \cdot & \cdot & 0 & 0 & -U & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & -2T & 0 & 0 & \cdot & \cdot & \cdot & -U \\ \hline -U & 0 & \cdot & \cdot & \cdot & 0 & -2S & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & -U & \cdot & \cdot & \cdot & 0 & 0 & -2S & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & -U & 0 & 0 & \cdot & \cdot & \cdot & -2S \end{array}\right]\underline{x}^*,$$

and hence,

$$\begin{aligned} M(T,S,U) &= (1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}(2\pi)^{-n} \\ &\quad \times\int\ldots\int \exp[(-1/2)\underline{x}^{*\prime}\underline{A}\ \underline{x}^*]dx_1\ldots dy_n, \end{aligned} \qquad (2.1.10)$$

where $\underset{\sim}{A}$ is the partitioned matrix,

$$\left[\begin{array}{c|c} \underset{\sim}{Q}_1 & \underset{\sim}{Q}_3 \\ \hline \underset{\sim}{Q}_3 & \underset{\sim}{Q}_2 \end{array}\right] ,$$

$$\text{with } \underset{\sim}{Q}_1 = \begin{bmatrix} 1-2T & -\rho_1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -\rho_1 & 1+\rho_1^2-2T & -\rho_1 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & -\rho_1 & 1+\rho_1^2-2T & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1+\rho_1^2-2T & -\rho_1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -\rho_1 & 1-2T \end{bmatrix}$$

,(2.1.11)

$\underset{\sim}{Q}_2$ identical to Q_1 with T and ρ_1 replaced by S and ρ_2, respectively, (2.1.12)

and $\underset{\sim}{Q}_3 = -U\underset{\sim}{I}$, where $\underset{\sim}{I}$ is the n×n identity matrix, (2.1.13)

Since $\underset{\sim}{A}$ is symmetric and, in general, non-singular, there exists an orthogonal transformation of $\mathbb{R}^{2n}$ which transforms $\underset{\sim}{A}$ into the identity matrix. The Jacobian of this transformation is $|\underset{\sim}{A}|^{-1/2}$ (see, for example, Aitken (1959))

$$\text{Hence, } M(T,S,U) = (1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}(2\pi)^{-n}|\underset{\sim}{A}|^{-1/2}$$

$$\times\int\ldots\int\exp[-1/2))(\Sigma x_i^2+\Sigma y_i^2)]dx_1\ldots dy_n$$

$$= (1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}/|\underset{\sim}{A}|^{1/2} . \qquad (2.1.14)$$

The joint moment generating function of C^*, D^*, and E^* is

$$M^*(T,S,U) = (1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}(2\pi)^{-n}$$

$$\times\int\ldots\int \exp[TC^*+SD^*+UE^*-(1/2)\underline{x}^{*\prime}\ \hat{\underline{A}}\ \underline{x}^*]\,dx_1\ldots dy_n,$$

where by equations (2.1.8),

$$TC^* = [\Sigma x_i^2-(1/n)(\ \Sigma x_i)^2]$$

$$=T[[\sum_i x_i^2-(1/n)(\sum_i\sum_j x_i x_j)]$$

$$= T[\sum_i (1-1/n)x_i^2+\sum_{i\neq j}(-1/n)x_i x_j],$$

$$SD^* = S[\sum_i (1-1/n)y_i^2+\sum_{i\neq j}(-1/n)y_i y_j],$$

and $UE^* = U[\sum_i (1-1/n)x_i y_i+\sum_{i\neq j}(-1/n)x_i y_j].$

Hence, using the same procedure as in the previous case,

we obtain $M^*(T,S,U) = (1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}/|\underline{A}^*|^{1/2}$, (2.1.15)

where $\underline{A}^*$ is the $2n\times 2n$ partitioned matrix,

$$\underline{A}^* = \left[\begin{array}{c|c} \underline{Q}_1^* & \underline{Q}_3^* \\ \hline \underline{Q}_3^* & \underline{Q}_2^* \end{array}\right],$$

$$\underset{\sim}{Q}_1^* = \begin{bmatrix} a+\alpha & -\rho_1+\alpha & \alpha & \cdot & \cdot & \cdot & \alpha & \alpha \\ -\rho_1+\alpha & b+\alpha & -\rho_1+\alpha & \cdot & \cdot & \cdot & \alpha & \alpha \\ \alpha & -\rho_1+\alpha & b+\alpha & \cdot & \cdot & \cdot & \alpha & \alpha \\ \vdots & \vdots & \vdots & \ddots & & & \vdots & \vdots \\ \alpha & \alpha & \alpha & \cdot & \cdot & \cdot & b+\alpha & -\rho_1+\alpha \\ \alpha & \alpha & \alpha & \cdot & \cdot & \cdot & -\rho_1+\alpha & a+\alpha \end{bmatrix}, \quad (2.1.16)$$

$$\underset{\sim}{Q}_2^* = \begin{bmatrix} c+\beta & -\rho_2+\beta & \beta & \cdot & \cdot & \cdot & \beta & \beta \\ -\rho_2+\beta & d+\beta & -\rho_2+\beta & \cdot & \cdot & \cdot & \beta & \beta \\ \beta & -\rho_2+\beta & d+\beta & \cdot & \cdot & \cdot & \beta & \beta \\ \vdots & \vdots & \vdots & \ddots & & & \vdots & \vdots \\ \beta & \beta & \beta & \cdot & \cdot & \cdot & d+\beta & -\rho_2+\beta \\ \beta & \beta & \beta & \cdot & \cdot & \cdot & -\rho_2+\beta & c+\beta \end{bmatrix}, \quad (2.1.17)$$

and

$$\underset{\sim}{Q}_3^* = \begin{bmatrix} -U+\gamma & \gamma & \gamma & \cdot & \cdot & \cdot & \gamma & \gamma \\ \gamma & -U+\gamma & \gamma & \cdot & \cdot & \cdot & \gamma & \gamma \\ \gamma & \gamma & -U+\gamma & \cdot & \cdot & \cdot & \gamma & \gamma \\ \vdots & \vdots & \vdots & \ddots & & & \vdots & \vdots \\ \gamma & \gamma & \gamma & \cdot & \cdot & \cdot & -U+\gamma & \gamma \\ \gamma & \gamma & \gamma & \cdot & \cdot & \cdot & \gamma & -U+\gamma \end{bmatrix}, \quad (2.1.18)$$

with $a = 1-2T, \quad b = 1+\rho_1^2-2T,$

$$c = 1-2S, \quad d = 1+\rho_2^2-2S, \quad (2.1.19)$$

$\alpha = 2T/n, \beta = 2s/n,$ and $\gamma = U/n.$

2.2 Evaluation of the Determinants

Because the determinants, $|\underline{A}|$ and $|\underline{A}^*|$, are difficult to evaluate exactly, an asymptotic result, first derived by McGregor (1959), is used. As the general result is well described by McGregor (1959) and by Bielenstein (1963), the method is demonstrated here for the special case of approximating $|\underline{A}|$. Later, $|\underline{A}^*|$ will be approximated by reducing it to a scalar multiplied by $|\underline{A}|$.

First, observe that

$$\underline{A} = \left[\begin{array}{c|c} \underline{Q}_1 & \underline{Q}_3 \\ \hline \underline{Q}_3 & \underline{Q}_2 \end{array}\right] = \left[\begin{array}{c|c} \underline{Q}_1 & \underline{0} \\ \hline \underline{0} & \underline{Q}_2 \end{array}\right] + \left[\begin{array}{c|c} \underline{0} & \underline{Q}_3 \\ \hline \underline{Q}_3 & \underline{0} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \underline{Q}_1 & \underline{0} \\ \hline \underline{0} & \underline{Q}_2 \end{array}\right] \left\{ \left[\begin{array}{c|c} \underline{I} & \underline{0} \\ \hline \underline{0} & \underline{I} \end{array}\right] + \left[\begin{array}{c|c} \underline{Q}^{-1} & \underline{0} \\ \hline \underline{0} & \underline{Q}_2^{-1} \end{array}\right] \left[\begin{array}{c|c} \underline{0} & \underline{Q}_3 \\ \hline \underline{Q}_3 & \underline{0} \end{array}\right] \right\}$$

$$= \left[\begin{array}{c|c} \underline{Q}_1 & \underline{0} \\ \hline \underline{0} & \underline{Q}_2 \end{array}\right] \times \left[\begin{array}{c|c} \underline{I} & \underline{Q}_1^{-1}\underline{Q}_3 \\ \hline \underline{Q}_2^{-1}\underline{Q} & \underline{I} \end{array}\right] .$$

$$\text{Hence, } |\underline{A}| = |\underline{Q}_1|\,|\underline{Q}_2| \left|\begin{array}{c|c} \underline{I} & \underline{Q}_1^{-1}\underline{Q}_3 \\ \hline \underline{Q}_2^{-1}\underline{Q}_3 & \underline{I} \end{array}\right| . \tag{2.2.1}$$

$$\text{But } \left|\begin{array}{c|c} \underline{I} & \underline{Q}_1^{-1}\underline{Q}_3 \\ \hline \underline{Q}_2^{-1}\underline{Q}_3 & \underline{I} \end{array}\right| = \frac{1}{|\underline{Q}_2^{-1}\underline{Q}_3|} \left|\begin{array}{c|c} \underline{I} & \underline{Q}_1^{-1}\underline{Q}_3 \\ \hline \underline{Q}_2^{-1}\underline{Q}_3 & \underline{I} \end{array}\right| |\underline{Q}_2^{-1}\underline{Q}_3|$$

$$= \frac{1}{|\underline{Q}_2^{-1}\underline{Q}_3|} \left|\begin{array}{c|c} \underline{I} & \underline{Q}_1^{-1}\underline{Q}_3 \\ \hline \underline{Q}_2^{-1}\underline{Q}_3 & \underline{I} \end{array}\right| \left|\begin{array}{c|c} \underline{I} & \underline{0} \\ \hline \underline{0} & \underline{Q}_2^{-1}\underline{Q}_3 \end{array}\right|$$

$$= \frac{1}{|\underline{Q}_2^{-1}\underline{Q}_3|} \left| \begin{array}{c|c} \underline{I} & \underline{Q}_1^{-1}\underline{Q}_3\underline{Q}_2^{-1}\underline{Q}_3 \\ \hline \underline{Q}_2^{-1}\underline{Q}_3 & \underline{Q}_2^{-1}\underline{Q}_3 \end{array} \right|$$

$$= \frac{1}{|\underline{Q}_2^{-1}\underline{Q}_3|} \left| \begin{array}{c|c} \underline{I}-\underline{Q}_1^{-1}\underline{Q}_3\underline{Q}_2^{-1}\underline{Q}_3 & \underline{Q}_1^{-1}\underline{Q}_3\underline{Q}_2^{-1}\underline{Q}_3 \\ \hline \underline{0} & \underline{Q}_2^{-1}\underline{Q}_3 \end{array} \right|$$

$$= |\underline{I} - \underline{Q}_1^{-1}\underline{Q}_3\underline{Q}_2^{-1}\underline{Q}_3|. \tag{2.2.2}$$

Furthermore, $\underline{Q}_1, \underline{Q}_2$, and $\underline{Q}_3$ are symmetric, and products of them are also symmetric and hence commutative.
Therefore, by (2.2.1) and (2.2.2),

$$\begin{aligned} |\underline{A}| &= |\underline{Q}_1\underline{Q}_2|\,|\underline{I} - \underline{Q}_1^{-1}\underline{Q}_3\underline{Q}_2^{-1}\underline{Q}_3| \\ &= |\underline{Q}_1\underline{Q}_2 - \underline{Q}_1\underline{Q}_2\underline{Q}_1^{-1}\underline{Q}_3\underline{Q}_2^{-1}\underline{Q}_3| \\ &= |\underline{Q}_1\underline{Q}_2 - \underline{Q}_3^2|. \end{aligned} \tag{2.2.3}$$

Using the notation of Bielenstein (1963), (2.2.3), (2.1.11), (2.1.12), and (2.1.13), yield

$$|\underline{A}| = \rho_1^n \rho_1^n |\underline{D}_n^{(0)}|$$

$$= \rho_1^n \rho_2^n \left| \begin{array}{cccccccccccc} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & x_1 & x_2 & x_1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & x_1 & x_2 & x_1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & x_1 & x_2 & x_1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \beta_{24} & \beta_{23} & \beta_{22} & \beta_{21} \\ 0 & 0 & 0 & 0 & 0 & 0 & & 0 & \beta_{14} & \beta_{13} & \beta_{12} & \beta_{11} \end{array} \right|$$

$$= \rho_1^n \rho_2^n \left| \begin{array}{c:c} \underline{B}_{2,4}^{(0)} & \underline{0}_{2,n-4} \\ \hdashline \multicolumn{2}{c}{\underline{C}_{n-4,n}} \\ \hdashline \underline{0}_{2,n-4} & \underline{H}_{2,4} \end{array} \right| , \tag{2.2.4}$$

where $x_1 = -\dfrac{1+\rho_1^2-2T}{\rho_1} - \dfrac{1+\rho_2^2-2S}{\rho_2}$,

$$x_2 = 2+ \frac{(1+\rho_1^2-2T)(1+\rho_2^2-2S)}{\rho_1\rho_2} - \frac{U^2}{\rho_1\rho_2} ,$$

$$\beta_{11} = \frac{(1-2T)(1-2S)}{\rho_1\rho_2} + 1 - \frac{U^2}{\rho_1\rho_2} , \tag{2.2.5}$$

$$\beta_{12} = -\frac{1-2T}{\rho_1} - \frac{1+\rho_2^2-2S}{\rho_2} ,$$

$$\beta_{21} = -\frac{1-2S}{\rho_2} - \frac{1+\rho_1^2-2T}{\rho_1} ,$$

$\beta_{13} = \beta_{24} = 1$, $\beta_{14} = 0$, $\beta_{22} = x_2$, and $\beta_{23} = x_1$.

Multiplying $\underline{D}_n^{(0)}$ on the right by the triangular matrix,

$$\begin{bmatrix} 1 & -x_1 & -x_2 & -x_1 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ & 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ & & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ & & & \cdot & & & & & & \\ & (0) & & & & \cdot & & & & \\ & & & & & & & \cdot & & \\ & & & & & & & & & 1 \end{bmatrix} ,$$

whose determinant is 1, gives $\left|\underline{D}_n^{(0)}\right|$

$$
= \begin{vmatrix}
\beta_{11} & \beta_{12}-\beta_{11}x_1 & \beta_{13}-\beta_{11}x_2 & \beta_{14}-\beta_{11}x_1 & -\beta_{11} & 0\ldots 0 & 0 & 0 & 0 & 0 \\
\beta_{21} & \beta_{22}-\beta_{21}x_1 & \beta_{23}-\beta_{21}x_2 & \beta_{24}-\beta_{21}x_1 & -\beta_{21} & 0\ldots 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0\ldots 0 & 0 & 0 & 0 & 0 \\
0 & 1 & x_1 & x_2 & x_1 & 1\ldots 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0\ldots 1 & x_1 & x_2 & x_1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0\ldots 0 & \beta_{24} & \beta_{23} & \beta_{22} & \beta_{21} \\
0 & 0 & 0 & 0 & 0 & 0\ldots 0 & \beta_{14} & \beta_{13} & \beta_{12} & \beta_{11}
\end{vmatrix}
$$

$$
= \begin{vmatrix}
\beta_{12}-\beta_{11}x_1 & \beta_{13}-\beta_{11}x_2 & \beta_{14}-\beta_{11}x_1 & -\beta_{11} & 0\ldots 0 & 0 & 0 & 0 & 0 \\
\beta_{22}-\beta_{21}x_1 & \beta_{23}-\beta_{21}x_2 & \beta_{24}-\beta_{21}x_1 & -\beta_{21} & 0\ldots 0 & 0 & 0 & 0 & 0 \\
1 & x_1 & x_2 & x_1 & 1\ldots 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0\ldots 1 & x_1 & x_2 & x_1 & 1 \\
0 & 0 & 0 & 0 & 0\ldots 0 & \beta_{24} & \beta_{23} & \beta_{22} & \beta_{21} \\
0 & 0 & 0 & 0 & 0\ldots 0 & \beta_{14} & \beta_{13} & \beta_{12} & \beta_{11}
\end{vmatrix}
$$

$$
= \left|\underline{D}_{n-1}^{(1)}\right| = \left[\begin{array}{c|c}
\underline{B}_{2,4}^{(1)} & \underline{0}_{2,n-4-1} \\
\hline
\multicolumn{2}{c}{\underline{C}_{n-4-1,n-1}} \\
\hline
\underline{0}_{2,n-4-1} & \underline{H}_{2,4}
\end{array}\right] .
$$

Note that $\underline{B}_{2,4}^{(1)} = \underline{B}_{2,4}^{(0)} \, \underline{T}_{4,4}$,

where $$\underset{=}{T}_{4,4} = \begin{bmatrix} -x_1 & -x_2 & -x_1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} , \tag{2.2.6}$$

and that $\underset{=}{C}^{(1)}_{n-5,n-1}$ has the same circular form as $\underset{=}{C}^{(0)}_{n-4,n}$. This reduction process can be repeated (n-4) times; that is until the matrix, $\underset{=}{C}^{(k)}_{n-4-k,n-k}$, has no rows. At each stage,

$\underset{=}{B}^{(k)}_{2,4} = \underset{=}{B}^{(k-1)}_{2,4} \underset{=}{T}_{4,4}$, and $\underset{=}{H}_{2,4}$ is untouched.

Hence, $$|\underset{=}{D}^{(0)}_n| = |\underset{=}{D}^{(n-4)}_4| = \begin{vmatrix} \underset{=}{B}^{(n-4)}_{2,4} \\ \hline \underset{=}{H}_{2,4} \end{vmatrix} = \begin{vmatrix} \underset{=}{B}^{(0)}_{2,4} \,|\, \underset{=}{T}^{n-4}_{4,4} \\ \hline \underset{=}{H}_{2,4} \end{vmatrix} . \tag{2.2.7}$$

Henceforth, $\underset{=}{B}^{(0)}_{2,4}$, $\underset{=}{H}_{2,4}$, and $\underset{=}{T}_{4,4}$ will be denoted by $\underset{=}{B}$, $\underset{=}{H}$, and $\underset{=}{T}$, respectively.

To find an expression for $\underset{=}{T}^{n-4}$, first let p_1, p_2, p_3, and p_4 be the roots of the equation, $p^4+x_1p^3+x_2p^2+x_1p+1 = 0$.
(2.2.8)

By the symmetry of the coefficients, roots occur in pairs, ϕ_1, $1/\phi_1$, ϕ_2, $1/\phi_2$. Assume (as can be done in general) that $|\phi_j|<1$, for j=1,2, and that $p_1=\phi_1$, $p_2=1/\phi_1$, $p_3=\phi_2$, and $p_4=1/\phi_2$.
(2.2.9)

Define

$$\underset{-}{P}(k) = \begin{bmatrix} p_1^{k+3} & p_2^{k+3} & p_3^{k+3} & p_4^{k+3} \\ p_1^{k+2} & p_2^{k+2} & p_3^{k+2} & p_4^{k+2} \\ p_1^{k+1} & p_2^{k+1} & p_3^{k+1} & p_4^{k+1} \\ p_1^{k} & p_2^{k} & p_3^{k} & p_4^{k} \end{bmatrix} . \tag{2.2.10}$$

Then $\underline{T}\ \underline{P}(k) = \underline{P}(k+1)$, and $\underline{T} = \underline{T}\ \underline{P}(0)\ \underline{P}^{-1}(0) = \underline{P}(1)\ \underline{P}^{-1}(0)$.

It follows by induction that $\underline{T}^k = \underline{P}(k)\ \underline{P}^{-1}(0)$. (2.2.11)

$$\text{Define } \underline{\Pi} = \underline{P}^{-1}(0) = \begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} & \pi_{14} \\ \pi_{21} & \pi_{22} & \pi_{23} & \pi_{24} \\ \pi_{31} & \pi_{32} & \pi_{33} & \pi_{34} \\ \pi_{41} & \pi_{42} & \pi_{43} & \pi_{44} \end{bmatrix} . \qquad (2.2.12)$$

Then, from (2.2.7), (2.2.11), and (2.2.12),

$$\underline{D}_4^{(n-4)} = \left| \begin{array}{c} \underline{B}\ \underline{P}(n-4)\ \underline{\Pi} \\ \hline \underline{H} \end{array} \right| . \qquad (2.2.13)$$

A typical element of the upper partition is, using (2.2.9),

$$e_{st} = \sum_i \sum_j b_{sj}(0) p_i^{n-j} \pi_{it} \sim \sum_j b_{sj}(0)(1/\phi_1)^{n-j}\pi_{2t} + b_{sj}(0)(1/\phi_2)^{n-j}\pi_{4t}$$

where terms becoming exponentially small as n increases (terms depending on ϕ_i^{n-j}) have been omitted.

Hence,

$$\underline{B}\ \underline{P}(n-4)\ \underline{\Pi} \sim \underline{B} \begin{bmatrix} p_2^{n-1} & p_4^{n-1} \\ p_2^{n-2} & p_4^{n-2} \\ p_2^{n-3} & p_4^{n-3} \\ p_2^{n-4} & p_4^{n-4} \end{bmatrix} \begin{bmatrix} \pi_{21} & \pi_{22} & \pi_{23} & \pi_{24} \\ \pi_{41} & \pi_{42} & \pi_{43} & \pi_{44} \end{bmatrix} , \qquad (2.2.14)$$

$$\text{and } \underline{D}_4^{(n-4)} \sim \left[\begin{array}{c} \underline{B} \begin{bmatrix} p_2^{n-1} & p_4^{n-1} \\ p_2^{n-2} & p_4^{n-2} \\ p_2^{n-3} & p_4^{n-3} \\ p_2^{n-4} & p_4^{n-4} \end{bmatrix} \begin{bmatrix} \pi_{21} & \pi_{22} & \pi_{23} & \pi_{24} \\ \pi_{41} & \pi_{42} & \pi_{43} & \pi_{44} \end{bmatrix} \\ \hline \underline{H} \end{array} \right]$$

$$= \left|\begin{array}{c|c} \underline{B}\begin{bmatrix} p_2^{n-1} & p_4^{n-1} \\ p_2^{n-2} & p_4^{n-2} \\ p_2^{n-3} & p_4^{n-3} \\ p_2^{n-4} & p_4^{n-4} \end{bmatrix} & (0) \\ \hline (0) & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{array}\right| \times \left|\begin{array}{cccc} \pi_{21} & \pi_{22} & \pi_{23} & \pi_{24} \\ \pi_{41} & \pi_{42} & \pi_{43} & \pi_{44} \\ \hline & & & \\ & \underline{H} & & \\ & & & \end{array}\right|$$

$$= K_1^{(1)} K_2 . \tag{2.2.15}$$

Then,

$$K_1^{(1)} = (p_1p_3)^{n-4} \left|\begin{array}{c|c} \underline{B}\begin{bmatrix} p_2^3 & p_4^3 \\ p_2^2 & p_4^2 \\ p_2 & p_4 \\ 1 & 1 \end{bmatrix} & \underline{B}\begin{bmatrix} p_1^3 & p_3^3 \\ p_1^2 & p_3^2 \\ p_1 & p_3 \\ 1 & 1 \end{bmatrix} \\ \hline \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{array}\right|$$

$$= (p_1p_3)^{n-4} \left|\begin{array}{cccc} & \underline{B} & & \\ \hline \pi_{11} & \pi_{12} & \pi_{13} & \pi_{14} \\ \pi_{31} & \pi_{32} & \pi_{33} & \pi_{34} \end{array}\right| \times \left|\begin{array}{cccc} p_2^3 & p_4^3 & p_1^3 & p_3^3 \\ p_2^2 & p_4^2 & p_1^2 & p_3^2 \\ p_2 & p_4 & p_1 & p_3 \\ 1 & 1 & 1 & 1 \end{array}\right| .$$

But $\left|\begin{array}{cccc} p_2^3 & p_4^3 & p_1^3 & p_3^3 \\ p_2^2 & p_4^2 & p_1^2 & p_3^2 \\ p_2 & p_4 & p_1 & p_3 \\ 1 & 1 & 1 & 1 \end{array}\right| = - \prod_{1 \leq i<j \leq 4} (p_i-p_j)$ (see Aitken (1959)).

Thus, $K_1^{(1)} = -(p_2p_4)^{n-4}\ K_1 \prod_{1\leq i<j\leq 4} (p_i-p_j)$, (2.2.16)

$$\text{where } K_1 = \left| \begin{array}{cccc} & \underline{B} & & \\ \hline \pi_{11} & \pi_{12} & \pi_{13} & \pi_{14} \\ \pi_{31} & \pi_{32} & \pi_{33} & \pi_{34} \end{array} \right| . \qquad (2.2.17)$$

To evaluate K_1 and K_2, we need more information on the matrix, $\underline{\Pi}$. Let $N(s;t)$ = the sum of all distinct products of p_1, p_2, p_3, p_4, taken s at a time, excluding p_t. (2.2.18)

$$\text{Let } M(t) = \prod_{\substack{i=1 \\ i\neq t}}^{4} (p_t-p_i) . \qquad (2.2.19)$$

Then, it can be shown (see Aitken (1959)) that

$\pi_{ij} = (-1)^{j-1}N(j-1;i)/M(i)$, where $N(0;i) = 1$, (2.2.20)

$$\text{whence } K_1 = \frac{1}{M(1)M(3)} \left| \begin{array}{cccc} & \underline{B} & & \\ \hline 1 & -N(1;1) & N(2;1) & -N(3;1) \\ 1 & -N(1;3) & N(2;3) & -N(3;3) \end{array} \right| . \qquad (2.2.21)$$

Therefore, using (2.2.5), (2.2.18), (2.2.19), and (2.2.21),

$$K_1 = \left((p_2-p_1)(p_3-p_1)(p_4-p_1)(p_1-p_3)(p_2-p_3)(p_4-p_3)\right)^{-1}$$

$$\left| \begin{array}{cccc} \beta_{11} & \beta_{12} & 1 & 0 \\ \beta_{21} & x_2 & x_1 & 1 \\ 1 & -(p_2+p_3+p_4) & p_2p_3+p_2p_4+p_3p_4 & p_2p_3p_4 \\ 1 & -(p_1+p_2+p_4) & p_1p_2+p_1p_4+p_2p_4 & p_1p_2p_4 \end{array} \right| . \qquad (2.2.22)$$

Using (2.2.9), this last determinant becomes

$$\frac{\phi_2-\phi_1}{(\phi_1\phi_2)^2}\begin{vmatrix} \beta_{11} & \beta_{12} & 1 & 0 \\ \beta_{21} & x_2 & x_1 & 1 \\ \phi_1\phi_2 & -(\phi_2+\phi_1\phi_2^2+\phi_1) & \phi_1\phi_2+\phi_2^2+1 & \phi_2 \\ 0 & \phi_1\phi_2 & -(\phi_1+\phi_2) & 1 \end{vmatrix}$$

$$= \frac{\phi_2-\phi_1}{(\phi_1\phi_2)^2}\begin{vmatrix} \beta_{11} & \beta_{12} & 1 & 0 \\ \beta_{21} & x_2 & x_1 & 1 \\ \phi_1\phi_2 & -(\phi_1+\phi_2) & 1 & 0 \\ 0 & \phi_1\phi_2 & -(\phi_1+\phi_2) & 1 \end{vmatrix} . \qquad (2.2.23)$$

Combining (2.2.22), and (2.2.23), replacing p's with ϕ's,

$$K_1 = \frac{1}{(\phi_1-1/\phi_1)(\phi_2-\phi_1)(\phi_1-1/\phi_2)(\phi_2-1/\phi_1)(\phi_2-1/\phi_2)(\phi_1\phi_2)^2}$$

$$\times \begin{vmatrix} \beta_{11} & \beta_{12} & 1 & 0 \\ \beta_{21} & x_2 & x_1 & 1 \\ \phi_1\phi_2 & -(\phi_1+\phi_2) & 1 & 0 \\ 0 & \phi_1\phi_2 & -(\phi_1+\phi_2) & 1 \end{vmatrix} . \qquad (2.2.24)$$

Similarly,

$$K_2 = \frac{1}{(\phi_1-1/\phi_1)(\phi_2-1/\phi_1)(\phi_1-1/\phi_2)(1/\phi_2-1/\phi_1)(\phi_2-1/\phi_2)}$$

$$\times \begin{vmatrix} 1 & -(\phi_1+\phi_2) & \phi_1\phi_2 & 0 \\ 0 & 1 & -(\phi_1+\phi_2) & \phi_1\phi_2 \\ 1 & x_1 & x_2 & \beta_{21} \\ 0 & 1 & \beta_{12} & \beta_{11} \end{vmatrix} . \qquad (2.2.25)$$

Making the substitutions, $a_1 = -(\phi_1+\phi_2)$, and $a_2 = \phi_1\phi_2$, (2.2.26)

$$(\phi_1-1/\phi_1)(\phi_2-1/\phi_2) = \phi_1\phi_2-(1/\phi_1\phi_2)(\phi_1^2+\phi_2^2+2\phi_1\phi_2)+2+(1/\phi_1\phi_2)$$

$$= a_2-(a_1^2/a_2)+2+(1/a_2)$$

$$=(1/a_2)(a_2^2-a_1^2+2a_2+1)$$

$$=(1/a_2)[(a_2+1)^2-a_1^2]$$

$$=(1/a_2)(1+a_1+a_2)(1-a_1+a_2), \qquad (2.2.27)$$

and $(\phi_1-1/\phi_2)(\phi_2-1/\phi_1) = \phi_1\phi_2-2+(1/\phi_1\phi_2)$

$$= a_2-2+(1/a_2)$$

$$=(1/a_2)(1-a_2)^2 \quad . \qquad (2.2.28)$$

Using (2.2.4), (2.2.15), (2.2.16), (2.2.24), and (2.2.25), and simplifying,

$$|\underline{A}| \sim \frac{\rho_1^n\rho_2^n}{a_2^{n-4}(1-a_1+a_2)(1+a_1+a_2)(1-a_2)^2}$$

$$\times \begin{vmatrix} \beta_{11} & \beta_{12} & 1 & 0 \\ \beta_{21} & x_2 & x_1 & 1 \\ a_2 & a_1 & 1 & 0 \\ 0 & a_2 & a_1 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & a_1 & a_2 & 0 \\ 0 & 1 & a_1 & a_2 \\ 1 & x_1 & x_2 & \beta_{21} \\ 0 & 1 & \beta_{12} & \beta_{11} \end{vmatrix} \quad . \qquad (2.2.29)$$

Recall that ϕ_1, $1/\phi_1$, ϕ_2, $1/\phi_2$ are the roots of

$$\phi^4+x_1\phi^3+x_2\phi^2+x_1\phi+1 = 0,$$

and hence $\quad x_1 = -(\phi_1+1/\phi_1+\phi_2+1/\phi_2)$

$$= -(\phi_1+\phi_2)-(\phi_1+\phi_2)/(\phi_1\phi_2)$$

$$= a_1 + a_1/a_2$$

$$= a_1(1 + 1/a_2) ,$$

and $$x_2 = \phi_1\phi_2 + \phi_1/\phi_2 + \phi_2/\phi_1 + 2 + 1/\phi_1\phi_2$$

$$= \phi_1\phi_2 + \frac{\phi_1^2+\phi_2^2+2\phi_1\phi_2}{\phi_1\phi_2} + \frac{1}{\phi_1\phi_2}$$

$$= a_2 + (a_1^2/a_2) + (1/a_2)$$

$$= (1/a_2)(1+a_1^2+a_2^2) . \qquad (2.2.31)$$

Using (2.2.30) and (2.2.31) in evaluating the determinants of (2.2.29), we obtain

$$|\underline{A}| \sim \frac{(1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2} a_2^{n/2}(1-a_1+a_2)^{1/2}(1+a_1+a_2)^{1/2}(1-a_2)}{(\rho_1\rho_2)^{n/2}[a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]} . \qquad (2.2.32)$$

Finally, substituting (2.2.32) in (2.1.14), the joint moment generating function of C, D, and E is found to be

$$M(T,S,U)$$

$$\sim \frac{(1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2} a_2^{n/2}(1-a_1+a_2)^{1/2}(1+a_1+a_2)^{1/2}(1-a_2)}{(\rho_1\rho_2)^{n/2}[a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]} \qquad (2.2.33)$$

To evaluate the determinant, $|\underline{A}^*| = \left|\begin{array}{c|c} \underline{Q}_1^* & \underline{Q}_3^* \\ \hline \underline{Q}_3^* & \underline{Q}_2^* \end{array}\right|$, where $\underline{Q}_1^*$, $\underline{Q}_2^*$, and $\underline{Q}_3^*$ are given in equations (2.1.16), (2.1.17), and (2.1.18), respectively, the bordering technique of Aitken (1959) and Muir (1960), followed by a series of matrix multiplications will be used.

The bordering technique amounts to no more than the simple observation that $|\underline{C}_1| = |\underline{A}^*|$, where

$$\underline{C}_1 = \left[\begin{array}{c|ccccc|ccccc|c}
1 & \alpha & \alpha & \cdot & \cdot & \cdot\ \alpha & \gamma & \gamma & \cdot & \cdot & \cdot\ \gamma & 0 \\
\hline
0 & & & & & & & & & & & 0 \\
\vdots & & & \underline{Q}_1^* & & & & & \underline{Q}_3^* & & & \vdots \\
0 & & & & & & & & & & & 0 \\
\hline
0 & & & & & & & & & & & 0 \\
\vdots & & & \underline{Q}_3^* & & & & & \underline{Q}_2^* & & & \vdots \\
0 & & & & & & & & & & & 0 \\
\hline
0 & \gamma & \gamma & \cdot & \cdot & \cdot\ \gamma & \beta & \beta & \cdot & \cdot & \cdot\ \beta & 1
\end{array}\right]$$

$\underline{C}_1$ is then multiplied on the left by

$$\underline{I}_1 = \left[\begin{array}{cccccc|cccccc}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
-1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & -1 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & -1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}\right],$$

yielding

$$
\underline{C}_2 = \left[\begin{array}{c|ccccc|ccccc|c}
1 & \alpha & \cdot & \cdot & \cdot & \alpha & \gamma & \cdot & \cdot & \cdot & \gamma & 0 \\
\hline
-1 & & & & & & & & & & & 0 \\
\vdots & & & \underline{Q}_1 & & & & & \underline{Q}_3 & & & \vdots \\
-1 & & & & & & & & & & & 0 \\
\hline
0 & & & & & & & & & & & -1 \\
\vdots & & & \underline{Q}_3 & & & & & \underline{Q}_2 & & & \vdots \\
0 & & & & & & & & & & & -1 \\
\hline
0 & \gamma & \cdot & \cdot & \cdot & \gamma & \beta & \cdot & \cdot & \cdot & \beta & 1
\end{array}\right], \qquad (2.2.36)
$$

and $|\underline{C}_2| = |\underline{I}_1||\underline{C}_1| = |\underline{C}_1| = |\underline{A}^*|$,

where $\underline{Q}_1$, $\underline{Q}_2$, and $\underline{Q}_3$ are given in (2.1.11), (2.1.12), and (2.1.13), respectively.

To obtain more zeros on the border, $\underline{C}_2$ is multiplied on the left by

$$
\underline{I}_2 = \begin{bmatrix}
1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\
0 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & & & \cdot & \cdot \\
\cdot & \cdot & & \cdot & & \cdot & \cdot \\
\cdot & \cdot & & & \cdot & \cdot & \cdot \\
0 & 0 & \cdot & \cdot & \cdot & 1 & 0 \\
-\frac{\gamma}{\alpha} & 0 & \cdot & \cdot & \cdot & 0 & 1
\end{bmatrix},
$$

producing

$$|\underline{C}_3| = |\underline{I}_2\underline{C}_2| = \left|\left[\begin{array}{c|ccccc|ccccc|c}
1 & \alpha & . & . & . & \alpha & \gamma & . & . & . & \gamma & 0 \\ \hline
-1 & & & & & & & & & & & 0 \\
. & & & & & & & & & & & . \\
. & & & \underline{Q}_1 & & & & & \underline{Q}_3 & & & . \\
. & & & & & & & & & & & . \\
-1 & & & & & & & & & & & 0 \\ \hline
0 & & & & & & & & & & & -1 \\
. & & & & & & & & & & & . \\
. & & & \underline{Q}_3 & & & & & \underline{Q}_2 & & & . \\
. & & & & & & & & & & & . \\
0 & & & & & & & & & & & -1 \\ \hline
-\gamma/\alpha & 0 & . & . & . & 0 & \beta-\gamma^2/\alpha & . & . & . & \beta-\gamma^2/\alpha & 1
\end{array}\right]\right|, \tag{2.2.37}$$

Once more, multiplying $\underline{C}_3$ on the left, this time by

$$\underline{I}_3 = \left[\begin{array}{ccccccc}
1 & 0 & . & . & . & 0 & -\gamma(\beta-\gamma^2/\alpha)^{-1} \\
0 & 1 & . & . & . & 0 & 0 \\
. & . & . & & & . & . \\
. & . & & . & & . & . \\
. & . & & & . & . & . \\
0 & 0 & . & . & . & 1 & 0 \\
0 & 0 & . & . & . & 0 & 1
\end{array}\right],$$

we obtain

$$|\underline{C}_4| = |\underline{I}_3\underline{C}_3| = \left|\begin{array}{c|ccccc|ccccc|c}
y_3 & \alpha & . & . & . & \alpha & 0 & . & . & . & 0 & y_4 \\ \hline
-1 & & & & & & & & & & & 0 \\
. & & & & & & & & & & & . \\
. & & & \underline{Q}_1 & & & & & \underline{Q}_3 & & & . \\
. & & & & & & & & & & & . \\
-1 & & & & & & & & & & & 0 \\ \hline
0 & & & & & & & & & & & -1 \\
. & & & & & & & & & & & . \\
. & & & \underline{Q}_3 & & & & & \underline{Q}_2 & & & . \\
. & & & & & & & & & & & . \\
0 & & & & & & & & & & & -1 \\ \hline
y_1 & 0 & . & . & . & 0 & y_2 & . & . & . & y_2 & 1
\end{array}\right|, \tag{2.2.38}$$

where $y_1 = -\gamma/\alpha$, $y_2 = \beta-\gamma^2/\alpha$,

$$y_3 = \frac{\alpha\beta}{\alpha\beta-\gamma^2} \text{ , and } y_4 = \frac{-\gamma}{\beta-\gamma^2/\alpha} . \tag{2.2.39}$$

To obtain more zeros in the first and last rows of $\underset{\sim}{C}_4$, let λ_1 and λ_2 satisfy the equations,

$$\alpha - \lambda_1(b-2\rho_1) - \lambda_2(-U) = 0 \quad ,$$

and $$-\lambda_1(-U) - \lambda_2(d-2\rho_2) = 0 \quad .$$

That is, let

$$\lambda_1 = \frac{\alpha(d-2\rho_2)}{(b-2\rho_1)(d-2\rho_2) - U^2}$$

and $$\lambda_2 = \frac{\alpha U}{(b-2\rho_1)(d-2\rho_2) - U^2} \quad .$$

Furthermore, let λ_3 and λ_4 satisfy the equations,

$$y_2 - \lambda_3(d-2\rho_2) - \lambda_4(-U) = 0 \quad ,$$

and $$-\lambda_3(-U) - \lambda_4(b-2\rho_1) = 0 \quad .$$

That is, let $$\lambda_3 = \frac{y_2(b-2\rho_1)}{(b-2\rho_1)(d-2\rho_2) - U^2} \quad ,$$

and $$\lambda_4 = \frac{y_2 U}{(b-2\rho_1)(d-2\rho_2) - U^2} \quad .$$

Then, $\underset{\sim}{C}_4$ is multiplied on the left by

$$\underline{I}_4 = \begin{bmatrix} 1 & -\lambda_1 & -\lambda_1 & \cdot & \cdot & \cdot & -\lambda_1 & -\lambda_2 & \cdot & \cdot & \cdot & -\lambda_2 & -\lambda_2 & 0 \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot & & & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot & & & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot & & & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 \\ 0 & -\lambda_4 & -\lambda_4 & \cdot & \cdot & \cdot & -\lambda_4 & -\lambda_3 & \cdot & \cdot & \cdot & -\lambda_3 & -\lambda_3 & 1 \end{bmatrix},$$

yielding $|\underline{C}_5| = |\underline{I}_4\underline{C}_4| = [(b-2\rho_1)(d-2\rho_2) - U^2]^{-2}$

$$\times \left| \begin{array}{c|cccc|cccc|c} y_5 & y_6 & 0 & \cdots & y_6 & y_7 & 0 & \cdots & y_7 & y_8 \\ \hline -1 & & & & & & & & & 0 \\ \vdots & & \underline{Q}_1 & & & & \underline{Q}_3 & & & \vdots \\ -1 & & & & & & & & & 0 \\ \hline 0 & & & & & & & & & -1 \\ \vdots & & \underline{Q}_3 & & & & \underline{Q}_2 & & & \vdots \\ 0 & & & & & & & & & -1 \\ \hline y_9 & y_{10} & 0 & \cdots & y_{10} & y_{11} & 0 & \cdots & y_{11} & y_{12} \end{array} \right| , \quad (2.2.39)$$

where $[(b-2\rho_1)(d-2\rho_2) - U^2]^{-1}$ has been factored from the first and last rows, and where

$$y_5 = [(b-2\rho_1)(d-2\rho_2) - U^2][y_3-n\lambda_1]$$

$$= y_3[(b-2\rho_1)(d-2\rho_2) - U^2] + n\alpha(d-2\rho_2) ,$$

$$y_6 = [(b-2\rho_1)(d-2\rho_2) - U^2][\alpha - \lambda_1(a-\rho_1) + \lambda_2 U]$$

$$= [(b-2\rho_1)(d-2\rho_2) - U^2] - \alpha(d-2\rho_2)(a-\rho_1) + \alpha U$$

$$= \alpha(d-2\rho_2)[b-2\rho_1-a+\rho_1]$$

$$= \alpha\rho_1(\rho_1-1)(d-2\rho_2) \ ,$$

$$y_7 = [(b-2\rho_1)(d-2\rho_2) - U^2][\lambda_1 U - \lambda_2(c-\rho_2)]$$

$$= \alpha(d-2\rho_2)U - \alpha U(c-\rho_2)$$

$$= \alpha U[d-2\rho_2-c+\rho_2]$$

$$= \alpha\rho_2(\rho_2-1)U \ ,$$

$$y_8 = [(b-2\rho_1)(d-2\rho_2) - U^2][y_4+n\lambda_2]$$

$$= y_4[(b-2\rho_1)(d-2\rho_2) - U^2] + n\alpha U \ ,$$

$$y_9 = [(b-2\rho_1)(d-2\rho_2) - U^2][n-\lambda_4 y_1]$$

$$= ny_2 U - y_1[(b-2\rho_1)(d-2\rho_2) - U^2] \ ,$$

$$y_{10} = [(b-2\rho_1)(d-2\rho_2) - U^2][-\lambda_4(a-\rho_1) - \lambda_3 U]$$

$$= -y_2 U(a-\rho_1) + y_2(b-2\rho_1)U$$

$$= y_2 U[b-2\rho_1-a+\rho_1]$$

$$= y_2\rho_1(\rho_1-1)U \ ,$$

$$y_{11} = [(b-2\rho_1)(d-2\rho_2) - U^2][\lambda_4 U - \lambda_3(c-\rho_2) + y_2]$$

$$= y_2 U^2 - y_2(b-2\rho_1)(c-\rho_2) + y_2[(b-2\rho_1)(d-2\rho_2) - U^2]$$

$$= y_2(b-2\rho_1)[d-2\rho_2-c+\rho_2]$$

$$= y_2\rho_2(\rho_2-1)(b-2\rho_1) \quad ,$$

and

$$y_{12} = [(b-2\rho_1)(d-2\rho_2) - U^2][n\lambda_3+1]$$

$$= ny_2(b-2\rho_1) + [(b-2\rho_1)(d-2\rho_2) - U^2] \; .$$

To obtain more zeros in the first and last columns of $\underset{\sim}{C}_5$, in equation (2.2.39), a similar technique is used.

Let μ_1 and μ_2 satisfy the equations,

$$-1 - \mu_1(b-2\rho_1) - \mu_2(-U) = 0 \quad ,$$

and
$$-\mu_1(-U) - \mu_2(d-2\rho_2) = 0 \quad .$$

Then
$$\mu_1 = \frac{-(d-2\rho_2)}{(b-2\rho_1)(d-2\rho_2) - U^2} \quad ,$$

$$\mu_2 = \frac{-U}{(b-2\rho_1)(d-2\rho_2) - U^2} \quad .$$

Furthermore, let μ_3 and μ_4 be the solutions of the equations,

$$-\mu_3(-U) - \mu_4(b-2\rho_1) = 0 \quad ,$$

and
$$-1 - \mu_3(d-2\rho_2) - \mu_4(-U) = 0 \quad .$$

Then
$$\mu_3 = \frac{-(b-2\rho_1)}{(b-2\rho_1)(d-2\rho_2) - U^2} \quad ,$$

and
$$\mu_4 = \frac{-U}{(b-2\rho_1)(d-2\rho_2) - U^2} \quad .$$

Finally, $\underline{C}_5$ is multiplied on the right by

$$\underline{I}_5 = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ -\mu_1 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & -\mu_4 \\ -\mu_1 & 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 & -\mu_4 \\ \vdots & \vdots & \vdots & & & & \vdots & \vdots & \vdots \\ -\mu_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & -\mu_4 \\ -\mu_2 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & -\mu_3 \\ \vdots & \vdots & \vdots & & & & \vdots & \vdots & \vdots \\ -\mu_2 & 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 & -\mu_3 \\ -\mu_2 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & -\mu_3 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 1 \end{bmatrix},$$

yielding $|\underline{C}_6| = |\underline{C}_5\underline{I}_5| = [(b-2\rho_1)(d-2\rho_2) - U^2]^{-4}$

$$\times \left| \begin{array}{c|ccccccc|ccccccc|c} w_1 & y_6 & 0 & \cdot & \cdot & \cdot & 0 & y_6 & y_7 & 0 & \cdot & \cdot & \cdot & 0 & y_7 & w_5 \\ \hline w_2 & & & & & & & & & & & & & & & w_6 \\ 0 & & & & & & & & & & & & & & & 0 \\ \vdots & \multicolumn{7}{c|}{\underline{Q}_1} & \multicolumn{7}{c|}{\underline{Q}_3} & \vdots \\ 0 & & & & & & & & & & & & & & & 0 \\ w_2 & & & & & & & & & & & & & & & w_6 \\ \hline w_3 & & & & & & & & & & & & & & & w_7 \\ 0 & & & & & & & & & & & & & & & 0 \\ \vdots & \multicolumn{7}{c|}{\underline{Q}_3} & \multicolumn{7}{c|}{\underline{Q}_2} & \vdots \\ 0 & & & & & & & & & & & & & & & 0 \\ w_3 & & & & & & & & & & & & & & & w_7 \\ \hline w_4 & y_{10} & 0 & \cdot & \cdot & \cdot & 0 & y_{10} & y_{11} & 0 & \cdot & \cdot & \cdot & 0 & y_{11} & w_8 \end{array} \right|, \quad (2.2.40)$$

where $[(b-2\rho_1)(d-2\rho_2) - U^2]^{-1}$ has been factored from the first and last columns, and where

$$w_1 = [(b-2\rho_1)(d-2\rho_2) - U^2][y_5 - \mu_1(2y_6) - \mu_2(2y_7)]$$

$$= y_5[(b-2\rho_1)(d-2\rho_2) - U^2] + 2[(d-2\rho_2)y_6 + Uy_7] ,$$

$$w_2 = [(b-2\rho_1)(d-2\rho_2) - U^2][-1 - \mu_1(a-\rho_1) + \mu_2 U]$$

$$= -[(b-2\rho_1)(d-2\rho_2) - U^2] + (d-2\rho_2)(a-\rho_1) - U^2$$

$$= -(d-2\rho_2)[b-2\rho_1-a+\rho_1]$$

$$= -\rho_1(\rho_1-1)(d-2\rho_2) ,$$

$$w_3 = [(b-2\rho_1)(d-2\rho_2) - U^2][\mu_1 U - \mu_2(c-\rho_2)]$$

$$= -(d-2\rho_2)U + (c-\rho_2)U$$

$$= -\rho_2(\rho_2-1)U ,$$

$$w_4 = [(b-2\rho_1)(d-2\rho_2) - U^2][y_9 - \mu_1(2y_{10}) - \mu_2(2y_{11})]$$

$$= y_9[(b-2\rho_1)(d-2\rho_2) - U^2] + 2[(d-2\rho_2)y_{10} + Uy_{11}] ,$$

$$w_5 = [(b-2\rho_1)(d-2\rho_2) - U^2][-\mu_4(2y_6) - \mu_3(2y_7) + y_8]$$

$$= 2[Uy_6 + (b-2\rho_1)y_7] + y_8[(b-2\rho_1)(d-2\rho_2) - U^2] ,$$

$$w_6 = [(b-2\rho_1)(d-2\rho_2) - U^2][-\mu_4(a-\rho_1) + \mu_3 U]$$

$$= U(a-\rho_1) - (b-2\rho_1)U$$

$$= -\rho_1(\rho_1-1)U ,$$

$$w_7 = [(b-2\rho_1)(d-2\rho_2) - U^2][\mu_4 U - \mu_3(c-\rho_2) - 1]$$

$$= -U^2 + (b-2\rho_1)(c-\rho_2) - [(b-2\rho_1)(d-2\rho_2) - U^2]$$

$$= -(b-2\rho_1)[d-2\rho_2-c+\rho_2]$$

$$= -\rho_2(\rho_2-1)(b-2\rho_1) \quad ,$$

and

$$w_8 = [(b-2\rho_1)(d-2\rho_2) - U^2][-\mu_4(2y_{10}) - \mu_3(2y_{11}) + y_{12}]$$

$$= 2[Uy_{10} + (b-2\rho_1)y_{11}] + y_{12}[(b-2\rho_1)(d-2\rho_2) - U^2] \quad .$$

The determinant in (2.2.40) may be evaluated asymptotically by observing the following:

Using equations (2.1.19),

$$y_6 = \alpha\rho_1(\rho_1-1)(d-2\rho_2) = O(1/n) \quad ,$$

$$y_7 = \alpha\rho_2(\rho_2-1)U = O(1/n) \quad ,$$

$$y_{10} = y_2\rho_1(\rho_1-1)U = (\beta-\gamma^2/\alpha)\rho_1(\rho_1-1)U = O(1/n) \quad ,$$

$$y_{11} = y_2\rho_2(\rho_2-1)(b-2\rho_1) = (\beta-\gamma^2/\alpha)\rho_2(\rho_2-1)(b-2\rho_1) = O(1/n) \quad ,$$

$$w_1 = y_5[(b-2\rho_1)(d-2\rho_2) - U^2] + 2[(d-2\rho_2)y_6 - Uy_7]$$

$$= y_3[(b-2\rho_1)(d-2\rho_2) - U^2]^2 + n\alpha(d-2\rho_2)[(b-2\rho_1)(d-2\rho_2) - U^2] + O(1/n)$$

$$= (4TS)(4TS-U^2)^{-1}[(b-2\rho_1)(d-2\rho_2) - U^2]^2 + 2T(d-2\rho_2)[(b-2\rho_1)(d-2\rho_2) - U^2] + O(1/n) \quad , \qquad (2.2.41)$$

$$w_2 = -\rho_1(\rho_1-1)(d-2\rho_2) \quad ,$$

$$w_3 = -\rho_2(\rho_2-1)U \quad ,$$

$$\begin{aligned}
w_4 &= y_9[(b-2\rho_1)(d-2\rho_2) - U^2] + 2[(d-2\rho_2)y_{10} + Uy_{11}] \\
&= ny_2U[(b-2\rho_1)(d-2\rho_2) - U^2] + y_1[(b-2\rho_1)(d-2\rho_2) - U^2]^2 + O(1/n) \\
&= (4TS-U^2)(2T)^{-1}U[(b-2\rho_1)(d-2\rho_2) - U^2] \\
&\qquad -(U/2T)[(b-2\rho_1)(d-2\rho_2) - U^2]^2 + O(1/n) \quad , \qquad (2.2.42)
\end{aligned}$$

$$\begin{aligned}
w_5 &= 2[Uy_6 - (b-2\rho_1)y_7] + y_8[(b-2\rho_1)(d-2\rho_2) - U^2] \\
&= O(1/n) + y_4[(b-2\rho_1)(d-2\rho_2) - U^2]^2 \\
&\qquad + n\alpha U[(b-2\rho_1)(d-2\rho_2) - U^2] \\
&= (-2TU)(4TS-U^2)^{-1}[(b-2\rho_1)(d-2\rho_2) - U^2]^2 \\
&\qquad + 2TU[(b-2\rho_1)(d-2\rho_2) - U^2] + O(1/n) \quad , \qquad (2.2.43)
\end{aligned}$$

$$w_6 = -\rho_1(\rho_1-1)U \quad ,$$

$$w_7 = -\rho_2(\rho_2-1)(b-2\rho_1) \quad ,$$

and

$$\begin{aligned}
w_8 &= 2[Uy_{10} + (b-2\rho_1)y_{11}] + y_{12}[(b-2\rho_1)(d-2\rho_2) - U^2] \\
&= O(1/n) + ny_2(b-2\rho_1)[(b-2\rho_1)(d-2\rho_2) - U^2] \\
&\qquad + [(b-2\rho_1)(d-2\rho_2) - U^2]^2 \\
&= (4TS-U^2)(2T)^{-1}(b-2\rho_1)[(b-2\rho_1)(d-2\rho_2) - U^2] \\
&\qquad + [(b-2\rho_1)(d-2\rho_2) - U^2]^2 + O(1/n) \quad . \qquad (2.2.44)
\end{aligned}$$

If we denote the determinant in equation (2.2.40) by B_1,

then $|\underline{A}^*| = [(b-2\rho_1)(d-2\rho_2) - U^2]^{-4} \; B_1$. (2.2.45)

Ignoring terms which are $O(1/n)$, B_1 may be expanded to the

simple form, $B_1 \sim w_1 B_2 + (-1)^{2n+1} w_5 B_3$;

that is $B_1 \sim w_1 B_2 - w_5 B_3$, (2.2.46)

where

$$B_2 = \left| \begin{array}{ccccc|ccccc|c} & & & & & & & & & & w_6 \\ & & & & & & & & & & 0 \\ & & \underline{Q}_1 & & & & & \underline{Q}_3 & & & \vdots \\ & & & & & & & & & & 0 \\ & & & & & & & & & & w_6 \\ \hline & & & & & & & & & & w_7 \\ & & & & & & & & & & 0 \\ & & \underline{Q}_3 & & & & & \underline{Q}_2 & & & \vdots \\ & & & & & & & & & & 0 \\ & & & & & & & & & & w_7 \\ \hline y_{10} & 0 & \cdots & 0 & y_{10} & y_{11} & 0 & \cdots & 0 & y_{11} & w_8 \end{array} \right| ,$$

and

$$
B_3 = \left|\begin{array}{c|c|c}
w_2 & & \\
0 & & \\
\vdots & \underset{\sim}{Q}_1 & \underset{\sim}{Q}_3 \\
0 & & \\
w_2 & & \\
\hline
w_3 & & \\
0 & & \\
\vdots & \underset{\sim}{Q}_3 & \underset{\sim}{Q}_2 \\
0 & & \\
w_3 & & \\
\hline
w_4 & y_{10} \; 0 \; \cdot \; \cdot \; \cdot \; 0 \; y_{10} & y_{11} \; 0 \; \cdot \; \cdot \; \cdot \; 0 \; y_{11}
\end{array}\right| .
$$

B_2 and B_3 may again be expanded. Ignoring terms of order n^{-1},

$$B_2 \sim w_8 |\underset{\sim}{A}| \, , \tag{2.2.47}$$

and

$$B_3 \sim w_4 |\underset{\sim}{A}| \, , \tag{2.2.48}$$

where $\underset{\sim}{A}$ is the matrix encountered in the 'known means' case.

Hence, using equations (2.2.45) through (2.2.48),

$$|\underset{\sim}{A}^*| = [(b-2\rho_1)(d-2\rho_2) - U^2]^{-4} \times [w_1w_8-w_5w_4] \; |\underset{\sim}{A}| \, . \tag{2.2.49}$$

Using relations (2.2.41) through (2.2.44), we may write

$$[w_1w_8-w_5w_4] = \{[4TS][4TS-U^2]^{-1}[(b-2\rho_1)(d-2\rho_2) -U^2]^2$$
$$+ 2T(d-2\rho_2)[(b-2\rho_1)(d-2\rho_2) - U^2]\}$$

(continued)

$$\times\{[4TS-U][2T]^{-1}(b-2\rho_1)[(b-2\rho_1)(d-2\rho_2) - U^2]$$

$$+ [(b-2\rho_1)(d-2\rho_2) - U^2]^2\}$$

$$- \{[-2TU][4TS-U^2]^{-1}[(b-2\rho_1)(d-2\rho_2) - U^2]^2$$

$$+ 2TU[(b-2\rho_1)(d-2\rho_2) - U^2]\}$$

$$\times\{[4TS-U^2][2T]^{-1}U[(b-2\rho_1)(d-2\rho_2) - U^2]$$

$$- (U/2T)[(b-2\rho_1)(d-2\rho_2) - U^2]^2\} + O(1/n)$$

$$= [(b-2\rho_1)(d-2\rho_2) - U^2]^2 \times \{2S(b-2\rho_1)$$

$$\times[(b-2\rho_1)(d-2\rho_2) - U^2]$$

$$+ [4TS][4TS-U^2]^{-1}[(b-2\rho_1)(d-2\rho_2) - U^2]^2$$

$$+ [4TS-U^2](d-2\rho_2)(b-2\rho_1) + 2T(d-2\rho_2)$$

$$\times[(b-2\rho_1)(d-2\rho_2) - U^2] + U^2[(b-2\rho_1)(d-2\rho_2) - U^2]$$

$$- U^2[4TS-U^2]^{-1}[(b-2\rho_1)(d-2\rho_2) - U^2]^2 - (4TS-U^2)U^2$$

$$+ U^2[(b-2\rho_1)(d-2\rho_2) - U^2]\} + O(1/n)$$

$$= [(b-2\rho_1)(d-2\rho_2) - U^2]^3$$

$$\times\{[4TS][4TS-U^2]^{-1}[(b-2\rho_1)(d-2\rho_2) - U^2] + 2S(b-2\rho_1)$$

$$+ 2U^2 - U^2[4TS-U^2]^{-1}[(b-2\rho_1)(d-2\rho_2) - U^2]+2T(d-2\rho_2)$$

$$+ [4TS-U^2]\} + O(1/n)$$

$$= [(b-2\rho_1)(d-2\rho_2) - U^2]^3$$

$$\times\{(b-2\rho_1)[d-2\rho_2+2S] + 2T[d-2\rho_2+2S]\} + O(1/n)$$

$$= [(b-2\rho_1)(d-2\rho_2) - U^2]^3\times[d-2\rho_2+2S][b-2\rho_1+2T] + O(1/n),$$

and, using equations (2.1.19),

$$w_1w_8-w_5w_4 = [(b-2\rho_1)(d-2\rho_2) - U^2]^3$$

$$\times[1+\rho_2^2-2S-2\rho_2+2S][1+\rho_1^2-2T-2\rho_1+2T] + O(1/n) .$$

Thus, (2.2.49) becomes

$$|\underline{A}^*| \sim [(b-2\rho_1)(d-2\rho_2) - U^2]^{-1}(1-\rho_1)^2(1-\rho_2)^2|\underline{A}| . \quad (2.2.50)$$

From (2.2.5), (2.2.31), and(2.2.32), we have the relations,

$$\left.\begin{aligned}&(1/a_2)(1+a_1^2+a_2^2) = 2 + [(1+\rho_1^2-2T)(1+\rho_2^2-2S)-U^2]/\rho_1\rho_2 ,\\ &\text{and}\\ &\quad(a_1/a_2)(1+a_2) = -(1+\rho_1^2-2T)/\rho_1 + (1+\rho_2^2-2S)/\rho_2 .\end{aligned}\right\}(2.2.51)$$

Now, $[(b-2\rho_1)(d-2\rho_2) - U^2] = bd + 2\rho_1\rho_2 - U^2 - 2(\rho_1 d+\rho_2 b) +2\rho_1\rho_2$,

so that, using (2.1.19),

$$[(b-2\rho_1)(d-2\rho_2) - U^2]$$

$$= \rho_1\rho_2\{(1+\rho_1^2-2T)(1+\rho_2^2-2S)/\rho_1\rho_2 + 2 - U^2/\rho_1\rho_2$$

$$- 2[(1+\rho_1^2-2T)/\rho_1 + (1+\rho_2^2-2S)/\rho_2] + 2\} ,$$

and, substituting relations (2.2.51),

$$[(b-2\rho_1)(d-2\rho_2) - U^2] = \rho_1\rho_2[(1/a_2)(1+a_1^2+a_2^2) + (2a_1/a_2)(1+a_2) + 2]$$

$$= (\rho_1\rho_2/a_2)[1+a_1^2+a_2^2+2a_1+2a_1a_2+2a_2]$$

$$= (\rho_1\rho_2/a_2)(1+a_1+a_2)^2 . \qquad (2.2.52)$$

Finally, substituting (2.2.33) and (2.2.52) in (2.2.50), we have

$$|\underline{A}^*| \sim \frac{(1-\rho_1)^2(1-\rho_2)^2 a_2}{\rho_1\rho_2(1+a_1+a_2)^2} \times \frac{\rho_1^n\rho_2^n[a_2(\beta_{11}-a_2) - a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]^2}{a_2^n(1-a_1+a_2)(1+a_1+a_2)(1-a_2)^2} ,$$

and, substituting for $|\underline{A}^*|$ in (2.1.20),

$$M^*(T,S,U) \sim \frac{(1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}}{(1-\rho_1)(1-\rho_2)} \times \frac{a_2^{(n-1)/2}}{(\rho_1\rho_2)^{(n-1)/2}}$$

$$\times \frac{(1-a_1+a_2)^{1/2}(1+a_1+a_2)^{3/2}(1-a_2)}{a_2(\beta_{11}-a_2) - a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)} , \qquad (2.2.53)$$

where the error in the approximation is $O(1/n)$.

2.3 A Method of Inversion

The following is a description of a technique for determining the joint density function of two ratios with common numerator.

Suppose the ratios are $b_1 = E/C$ and $b_2 = E/D$, where C and D are positive. The method used involves manipulating the joint moment generating function,

$$M(T,S,U) = E[\exp(TC+SD+UE)].$$

The transformation, $b_1 = E/C$, $b_2 = E/D$, $u = C+D$ is made, yielding $C = b_2u/(b_1+b_2)$, $D = b_1u/(b_1+b_2)$, $E = b_1b_2u/(b_1+b_2)$, with Jacobian, $b_1b_2u^2/(b_1+b_2)^3$.

Thus, if the joint density of C, D, and E is f(C,D,E), then that of b_1, b_2, and u is

$$f\left[\frac{b_2u}{b_1+b_2}, \frac{b_1u}{b_1+b_2}, \frac{b_1b_2u}{b_1+b_2}\right] \frac{b_1b_2u^2}{(b_1+b_2)^3} ,$$

and the joint density of b_1 and b_2 is

$$h(b_1,b_2) = \int_0^\infty f\left[\frac{b_2u}{b_1+b_2}, \frac{b_1u}{b_1+b_2}, \frac{b_1b_2u}{b_1+b_2}\right] \frac{b_1b_2u^2}{(b_1+b_2)^3} du .$$

But M(T,S,U) is the Fourier transform of f(C,D,E), and by the Fourier inversion theorem,

$$f(C,D,E) = (2\pi i)^{-3} \iiint M(T,S,U) \exp[-(TC+SD+UE)]dUdSdT .$$

Hence, $f\left[\dfrac{b_2u}{b_1+b_2}, \dfrac{b_1u}{b_1+b_2}, \dfrac{b_1b_2u}{b_1+b_2}\right]$

(continued)

$$= (2\pi i)^{-3} \iiint M(T,S,U) \exp - \left[\frac{Tb_2}{b_1+b_2} + \frac{Sb_1}{b_1+b_2} + \frac{Ub_1b_2}{b_1+b_2} \right] UdUdSdT \ , \tag{2.3.1}$$

where integrations are along the imaginary axes in the U, S, and T planes, or along allowable deformations of these paths.

Let $v = \dfrac{Tb_2}{b_1+b_2} + \dfrac{Sb_1}{b_1+b_2} + \dfrac{Ub_1b_2}{b_1+b_2}$.

Then $U = \dfrac{v(b_1+b_2)-Tb_2-Sb_1}{b_1b_2}$, $\quad dU = \dfrac{b_1+b_2}{b_1b_2} dv,$

and from (2.3.1),

$$f\left[\frac{b_2u}{b_1+b_2}, \frac{b_1u}{b_1+b_2}, \frac{b_1b_2u}{b_1+b_2}\right]$$

$$= (2\pi i)^{-3} \iiint M\left[T,S, \frac{v(b_1+b_2)-Tb_2-sb_1}{b_1b_2}\right] \exp(-vu) \ \frac{b_1+b_2}{b_1b_2} \ dvdSdT,$$

where integration with respect to v is along the transformed path in the v-plane.

Thus $\dfrac{b_1b_2}{(b_1+b_2)^3} \quad f\left[\dfrac{b_2u}{b_1+b_2}, \dfrac{b_1u}{b_1+b_2}, \dfrac{b_1b_2u}{b_1+b_2}\right]$

$$= (2\pi i)^{-3} \iiint M \ T, S, \ \frac{v(b_1+b_2)-Tb_2-Sb_1}{b_1b_2} \ [\exp(-vu)] (b_1+b_2)^{-2} dvdSdT. \tag{2.3.2}$$

Multiplying (2.3.2) by exp(vu) and integrating with respect to u, we obtain

$$\int_0^\infty \frac{b_1b_2}{(b_1+b_2)^3} \quad f\left[\frac{b_2u}{b_1+b_2}, \frac{b_1u}{b_1+b_2}, \frac{b_1b_2u}{b_1+b_2}\right] [\exp(vu)] du$$

$$= (b_1+b_2)^{-2}(2\pi i)^{-3}\int_0^\infty [\exp(vu)]\iiint M\left[T,S,\frac{v(b_1+b_2)-Tb_2-Sb_1}{b_1b_2}\right]\exp(-vu)$$

$$\times dvdSdTdu$$

$$=(b_1+b_2)^{-2}(2\pi i)^{-2}\iint\left[\int_0^\infty (2\pi i)^{-1}\int M\left[T,S,\frac{v(b_1+b_2)-Tb_2-Sb_1}{b_1b_2}\right]e^{-vu}\,dv\right.$$

$$\left.\times e^{vu}du\right]dSdT$$

$$=(b_1+b_2)^{-2}(2\pi i)^{-2}\iint M\left[T,S,\frac{v(b_1+b_2)-Tb_2-Sb_1}{b_1b_2}\right]dSdT \quad .$$

Finally, differentiating twice under the integral signs with respect to v and setting v = 0, we get

$$h(b_1,b_2) = \int_0^\infty \frac{b_1b_2}{(b_1+b_2)^3}\quad f\left[\frac{b_2u}{b_1+b_2},\frac{b_1u}{b_1+b_2},\frac{b_1b_2u}{b_1+b_2}\right]u^2du$$

$$=(b_1+b_2)^{-2}(2\pi i)^{-2}\iint \frac{\partial^2}{\partial v^2}\quad M\left[T,S,\frac{v(b_1+b_2)-Tb_2-Sb_2}{b_1b_2}\right]\Bigg|_{v=0}$$

$$\times\ dSdT \ . \qquad (2.3.3)$$

2.4 The Joint Density of the Two Regression Coefficients

(a) Known Means

Recall equation (2.2.33),

$$M(T,S,U) \sim \frac{(1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}\, a_2^{n/2}(1-a_1+a_2)^{1/2}(1+a_1+a_2)^{1/2}(1-a_2)}{(\rho_1\rho_2)^{n/2}\,[a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]}, \tag{2.4.1}$$

where
$$\left.\begin{aligned} \frac{1+a_1^2+a_2^2}{a_2} &= x_1 = \frac{(1+\rho_1^2-2T)(1+\rho_2^2-2S)}{\rho_1\rho_2} - \frac{U^2}{\rho_1\rho_2}, \\ \frac{a_1(1+a_2)}{a_2} &= -\frac{1+\rho_1^2-2T}{\rho_1} - \frac{1+\rho_2^2-2S}{\rho_2}, \end{aligned}\right\} \tag{2.4.2}$$

$$\left.\begin{aligned} \beta_{11} &= \frac{(1-2T)(1-2S)}{\rho_1\rho_2} + 1 - \frac{U^2}{\rho_1\rho_2} \\ \beta_{12} &= -\frac{1-2T}{\rho_1} - \frac{-1+\rho_2^2-2S}{\rho_2}, \\ \text{and}\quad \beta_{21} &= -\frac{1+\rho_1^2-2T}{\rho_1} - \frac{1-2S}{\rho_2}. \end{aligned}\right\} \tag{2.4.3}$$

Following the development of the last section, let

$$U = \frac{v(b_1+b_2)}{b_1b_2} - \frac{T}{b_1} - \frac{S}{b_2}, \tag{2.4.4}$$

where $b_1 = E/C$, and $b_2 = E/D$ are the regression coefficients. Then, by (2.4.2), a_1 and a_2 are implicit functions of T, S,

and v, and

$$\frac{\partial^2}{\partial v^2} M\left[T,S,\ \frac{v(b_1+b_2)-Tb_2-Sb_1}{b_1b_2}\right]$$

$$\sim \frac{(1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}(n/2)(n-2)a_2^{(n-4)/2}}{2(\rho_1\rho_2)^{n/2}}$$

$$\times\frac{(1-a_1+a_2)^{1/2}(1+a_1+a_2)^{1/2}(1-a_2)}{a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)}\quad\left[\frac{\partial a_2}{\partial v}\right]^2 ,$$

where terms of order atmost 1/n have been neglected. Therefore, using the inversion formula (2.3.3),

$$h(b_1,b_2) = \frac{(1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}(n/2)(n-2)}{2(\rho_1\rho_2)^{n/2}(b_1+b_2)^2(2\pi i)^2}$$

$$\times\iint a_2^{(n-4)/2}\ \frac{(1-a_1+a_2)^{1/2}(1+a_1+a_2)^{1/2}(1-a_2)}{a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)}$$

$$\times\left[\frac{\partial a_2}{\partial v}\right]^2\Bigg|_{v=0} dSdT + O(n^{-1}) \quad . \tag{2.4.5}$$

The double integral is evaluated using the following bivariate saddlepoint approximation (see De Bruijn (1961)):

$$(2\pi i)^{-2}\iint[\psi(z_1,z_2)]^k\phi(z_1,z_2)dz_2dz_1$$

$$\sim \phi(\hat{z}_1,\hat{z}_2)[\psi(\hat{z}_1,\hat{z}_2)]^{k+1}(2\pi k)^{-1}\ \left[\left[\frac{\partial^2\psi}{\partial z_1^2}(\hat{z}_1,\hat{z}_2)\right]\left[\frac{\partial^2\psi}{\partial z_2^2}(\hat{z}_1,\hat{z}_2)\right]\right.$$

$$\left.-\left[\frac{\partial\ \psi}{\partial z_1\partial z_2}(\hat{z}_1,\hat{z}_2)\right]^2\right]^{-1/2} , \tag{2.4.6}$$

where $\hat{z}_1$ and $\hat{z}_2$ are solutions of $\frac{\partial\psi}{\partial z_1}(z_1,z_2) = 0$, and $\frac{\partial\psi}{\partial z_1}(z_1,z_2) = 0$.

In (2.4.5), T and S correspond to z_1 and z_2 in (2.4.6) respectively, $a_2|_{v=0}$ to $\psi(z_1,z_2)$, $(n/2-2)$ to k, and

$$\left[\frac{(1-a_1+a_2)^{1/2}(1+a_1+a_2)^{1/2}(1-a_2)}{a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)} \left.\frac{\partial a_2}{\partial v}^2\right|_{v=0}\right] \text{ to } \phi(z_1,z_2) \text{ .}$$

The paths of integration in the T and S planes are taken as the lines of steepest descent of $a_2|_{v=0}$. If we denote $a_2|_{v=0}$ by z, then z is a function of T and S, and the paths of steepest descent are given by

$$\frac{\partial z}{\partial S}(\hat{T},\hat{S}) = \frac{\partial z}{\partial T}(\hat{T},\hat{S}) = 0 \text{ .}$$

Also denote $a_1|_{v=0}$ by ζ.

With this notation, eliminating ζ from equations (2.4.2) gives

$$\frac{1+z^2}{z} + \frac{z}{(1+z)^2}\left[\frac{1+\rho_1^2-2T}{\rho_1} + \frac{1+\rho_2^2-2S}{\rho_2}\right]^2$$

$$= 2 + \frac{(1+\rho_1^2-2T)(1+\rho_2^2-2S)}{\rho_1\rho_2} - \frac{1}{\rho_1\rho_2}\left[\frac{T}{b_1} + \frac{S}{b_2}\right]^2 \text{ .} \qquad (2.4.7)$$

Differentiating (2.4.7) implicitly with respect to T (holding S constant), and setting $\frac{\partial a_2}{\partial T} = 0$ yields

$$\frac{-4\hat{z}}{(1+\hat{z})^2\rho_1}\left[\frac{1+\rho_1^2-2\hat{T}}{\rho_1}+\frac{1+\rho_2^2-2\hat{S}}{\rho_2}\right]=\frac{-2(1+\rho_2^2-2\hat{S})}{\rho_1\rho_2}-\frac{2}{b_1\rho_1\rho_2}\left[\frac{\hat{T}}{b_1}+\frac{\hat{S}}{b_2}\right]$$

that is

$$\frac{2\hat{z}}{(1+\hat{z})^2\rho_1}\left[\frac{1+\rho_1^2-2\hat{T}}{\rho_1}+\frac{1+\rho_2^2-2\hat{S}}{\rho_2}\right]=\frac{(1+\rho_2^2-2\hat{S})}{\rho_1\rho_2}+\frac{1}{b_1\rho_1\rho_2}\left[\frac{\hat{T}}{b_1}+\frac{\hat{S}}{b_2}\right],\tag{2.4.8}$$

where $\hat{z} = z(T,S)$.

Similarly, differentiating a_2 with respect to S (holding T constant) and setting $\frac{\partial a_2}{\partial S} = 0$, we obtain

$$\frac{2\hat{z}}{(1+\hat{z})^2\rho_2}\left[\frac{1+\rho_1^2-2\hat{T}}{\rho_1}+\frac{1+\rho_2^2-2\hat{S}}{\rho_2}\right]=\frac{1+\rho_1^2-2\hat{T}}{\rho_1\rho_2}+\frac{1}{b_1\rho_1\rho_2}\left[\frac{\hat{T}}{b_1}+\frac{\hat{S}}{b_2}\right].\tag{2.4.9}$$

Equations (2.4.8) and (2.4.9) are now solved for the saddlepoint, $(\hat{T},\hat{S})$. From (2.4.8),

$$\hat{T}\left[\frac{-4\hat{z}}{(1+\hat{z})^2\rho_1^2}-\frac{1}{b_1^2\rho_1\rho_2}\right]=\frac{1+\rho_2^2-2\hat{S}}{\rho_1\rho_2}+\frac{\hat{S}}{b_1b_2\rho_1\rho_2}-\frac{2\hat{z}(1+\rho_1^2)}{(1+\hat{z})^2\rho_1^2}$$
$$-\frac{2\hat{z}(1+\rho_2^2-2\hat{S})}{(1+\hat{z})^2\rho_1\rho_2},$$

and hence,

$$\hat{T}=\left[\frac{-4\hat{z}}{(1+\hat{z})^2\rho_1^2}-\frac{1}{b_1^2\rho_1\rho_2}\right]^{-1}\left[\frac{-\hat{S}}{\rho_1\rho_2}\left[\frac{4\hat{z}}{(1+\hat{z})^2}+\frac{1}{b_1b_2}-2\right]\right.$$
$$\left.+\frac{2\hat{z}}{(1+\hat{z})^2\rho_1}\left[\frac{1+\rho_1^2}{\rho_1}+\frac{1+\rho_2^2}{\rho_2}\right]-\frac{1+\rho_2^2}{\rho_1\rho_2}\right].\tag{2.4.10}$$

Similarly, from (2.4.9),

$$\hat{S} = \left[\frac{4\hat{z}}{(1+\hat{z})^2\rho_2^2} + \frac{1}{b_2^2\rho_1\rho_2}\right]^{-1}\left[\frac{-\hat{T}}{\rho_1\rho_2}\left[\frac{4\hat{z}}{(1+\hat{z})^2} + \frac{1}{b_1 b_2} - 2\right]\right.$$

$$\left. + \frac{2\hat{z}}{(1+\hat{z})^2\rho_2}\left[\frac{1+\rho_1^2}{\rho_1} + \frac{1+\rho_2^2}{\rho_2}\right] - \frac{1+\rho_1^2}{\rho_1\rho_2}\right]. \qquad (2.4.11)$$

Substituting (2.4.10) in (2.4.11) gives

$$\hat{S} = \left[\frac{4\hat{z}}{(1+\hat{z})^2\rho_2^2} + \frac{1}{b_2^2\rho_1\rho_2}\right]^{-1}\left[\frac{2\hat{z}}{(1+\hat{z})^2\rho_2}\left(\frac{1+\rho_1^2}{\rho_1} + \frac{1+\rho_2^2}{\rho_2}\right) - \frac{1+\rho_1^2}{\rho_1\rho_2}\right.$$

$$- \frac{1}{\rho_1\rho_2}\left(\frac{4\hat{z}}{(1+\hat{z})^2} + \frac{1}{b_1 b_2} - 2\right) \times \left(\frac{4\hat{z}}{(1+\hat{z})^2\rho_1^2} + \frac{1}{b_1^2\rho_1\rho_2}\right)^{-1}$$

$$\left. \times \left[\frac{2\hat{z}}{(1+\hat{z})^2\rho_1}\left(\frac{1+\rho_1^2}{\rho_1} + \frac{1+\rho_2^2}{\rho_2}\right) - \frac{1+\rho_2^2}{\rho_1\rho_2} - \frac{\hat{S}}{\rho_1\rho_2}\left(\frac{4\hat{z}}{(1+\hat{z})^2} + \frac{1}{b_1 b_2} - 2\right)\right]\right] ,$$

which becomes

$$\hat{S} = \left[\left[\frac{2\hat{z}}{(1+\hat{z})^2\rho_2}\left(\frac{1+\rho_1^2}{\rho_1} + \frac{1+\rho_2^2}{\rho_2}\right) - \frac{1+\rho_1^2}{\rho_1\rho_2}\right]\left[\frac{4\hat{z}}{(1+\hat{z})^2\rho_1} + \frac{1}{b_1^2\rho_1\rho_2}\right]\right.$$

$$\left. - \frac{1}{\rho_1\rho_2}\left[\frac{4\hat{z}}{(1+\hat{z})^2} + \frac{1}{b_1 b_2} - 2\right]\left[\frac{2\hat{z}}{(1+\hat{z})^2\rho_1}\left(\frac{1+\rho_1^2}{\rho_1} + \frac{1+\rho_2^2}{\rho_2}\right) - \frac{(1+\rho_2)^2}{\rho_1\rho_2}\right]\right]$$

$$\times \left[\left[\frac{4\hat{z}}{(1+\hat{z})^2\rho_2^2} + \frac{1}{b_2^2\rho_1\rho_2}\right]\left[\frac{4\hat{z}}{(1+\hat{z})^2\rho_1^2} + \frac{1}{b_1^2\rho_1\rho_2}\right]\right.$$

$$\left. - \frac{1}{\rho_1^2\rho_2^2}\left[\frac{4\hat{z}}{(1+\hat{z})^2} + \frac{1}{b_1 b_2} - 2\right]^2\right]^{-1} .$$

The numerator of $\hat{S}$ is

$$\frac{8\hat{z}^2}{(1+\hat{z})^4\rho_1^2\rho_2}\left(\frac{1+\rho_1^2}{\rho_1}+\frac{1+\rho_2^2}{\rho_2}\right)+\frac{2\hat{z}}{(1+\hat{z})^2b_1^2\rho_1\rho_2}\left(\frac{1+\rho_1^2}{\rho_1}+\frac{1+\rho_2^2}{\rho_2}\right)$$

$$-\frac{4\hat{z}(1+\rho_1^2)}{(1+\hat{z})^2\rho_1^3\rho_2}-\frac{1+\rho_1^2}{b_1^2\rho_1^2\rho_2^2}-\frac{8\hat{z}^2}{(1+\hat{z})^4\rho_1^2\rho_2}\left(\frac{1+\rho_1^2}{\rho_1}+\frac{1+\rho_2^2}{\rho_2}\right)$$

$$-\frac{2\hat{z}}{(1+\hat{z})^2b_1b_2\rho_1^2\rho_2}\left(\frac{1+\rho_1^2}{\rho_1}+\frac{1+\rho_2^2}{\rho_2}\right)+\frac{4\hat{z}}{(1+\hat{z})^2\rho_1^2\rho_2}\left(\frac{1+\rho_1^2}{\rho_1}+\frac{1+\rho_2^2}{\rho_2}\right)$$

$$+\frac{4\hat{z}(1+\rho_2^2)}{(1+\hat{z})^2\rho_1^2\rho_2^2}+\frac{1+\rho_2^2}{b_1b_2\rho_1^2\rho_2^2}-\frac{2(1+\rho_2^2)}{\rho_1^2\rho_2^2}$$

$$= \left[(1+\hat{z})^2b_1^2b_2^2\rho_1^3\rho_2^3\right]^{-1}\left[2\hat{z}(1+\rho_1^2)b_2^2\rho_1\rho_2 + 2\hat{z}(1+\rho_2^2)b_2^2\rho_1^2\right.$$

$$- (1+\hat{z}^2+2\hat{z})(1+\rho_1^2)b_2^2\rho_1\rho_2 - 2\hat{z}(1+\rho_1^2)b_1b_2\rho_2^2 - 2\hat{z}(1+\rho_2^2)b_1b_2\rho_1\rho_2$$

$$+ 8\hat{z}(1+\rho_2^2)b_1^2b_2^2\rho_1\rho_2 + (1+\hat{z}^2+2\hat{z})(1+\rho_2^2)b_1b_2\rho_1\rho_2$$

$$\left.- 2(1+\hat{z}^2+2\hat{z})(1+\rho_2^2)b_1^2b_2^2\rho_1\rho_2\right]$$

$$= \left[(1+\hat{z}^2)b_1^2b_2^2\rho_1^3\rho_2^3\right]^{-1}\left[\, 2\hat{z}\left[(1+\rho_2^2)b_2^2\rho_1^2 + (1+\rho_1^2)b_1b_2\rho_2^2\right.\right.$$

$$\left. + 2(1+\rho_2^2)b_1^2b_2^2\rho_1\rho_2\right]$$

$$\left. - (1+\hat{z}^2)\rho_1\rho_2\left[(1+\rho_1^2)b_2^2 - (1+\rho_2^2)b_1b_2 + 2(1+\rho_2^2)b_1^2b_2^2\right]\right],$$

and the denominator of $\hat{S}$ is

$$\frac{16\hat{z}^2}{(1+\hat{z})^4\rho_1^2\rho_2^2} + \frac{4\hat{z}}{(1+\hat{z})^2 b_1^2\rho_1\rho_2^3} + \frac{4\hat{z}}{(1+\hat{z})^2 b_2^2\rho_1^3\rho_2} + \frac{1}{b_1^2 b_2^2\rho_1\rho_2}$$

$$- \frac{16\hat{z}^2}{(1+\hat{z})^4\rho_1^2\rho_2^2} - \frac{1}{b_1^2 b_2^2\rho_1^2\rho_2^2} - \frac{4}{\rho_1^2\rho_2^2} - \frac{8\hat{z}}{(1+\hat{z})^2 b_1 b_2\rho_1^2\rho_2^2}$$

$$+ \frac{16\hat{z}}{(1+\hat{z})^2\rho_1^2\rho_2^2} + \frac{4}{b_1 b_2\rho_1^2\rho_2^2}$$

$$= 4\left[(1+\hat{z})^2 b_1^2 b_2^2\rho_1^3\rho_2^3\right]^{-1}\left[\hat{z}b_2^2\rho_1^2 + \hat{z}b_1^2\rho_2^2 - (1+\hat{z}^2+2\hat{z})b_1^2 b_2^2\rho_1\rho_2 \right.$$
$$\left. - 2\hat{z}b_1 b_2\rho_1\rho_2 + 4\hat{z}b_1^2 b_2^2\rho_1\rho_2 + (1+\hat{z}^2+2\hat{z})b_1 b_2\rho_1\rho_2\right]$$

$$= 4\left[(1+\hat{z})^2 b_1^2 b_2^2\rho_1^3\rho_2^3\right]^{-1}\left[\hat{z}\left[b_1^2\rho_2^2 + b_2^2\rho_1^2 + 2b_1^2 b_2^2\rho_1\rho_2\right] \right.$$
$$\left. + (1+\hat{z}^2)b_1 b_2\rho_1\rho_2(1-b_1 b_2)\right].$$

Finally, the solution for $\hat{S}$ is obtained as

$$\hat{S} = (1/4)\left[\hat{z}(b_1^2\rho_2^2+b_2^2\rho_1^2+2b_1^2b_2^2\rho_1\rho_2) + (1+\hat{z}^2)b_1b_2\rho_1\rho_2(1-b_1b_2)\right]^{-1}$$
$$\times\left[2\hat{z}\left[b_2^2\rho_1^2(1+\rho_2^2) - b_1b_2\rho_2^2(1+\rho_1^2) + 2b_1^2b_2^2\rho_1\rho_2(1+\rho_2^2)\right]\right.$$
$$\left. - (1+\hat{z}^2)\rho_1\rho_2\left[b_2^2(1+\rho_1^2) - b_1b_2(1+\rho_2^2) + 2b_1^2b_2^2(1+\rho_2^2)\right]\right]. \quad (2.4.12)$$

By the symmetry of equations (2.4.9) and (2.4.10), the solution for $\hat{T}$ is obtained as

$$\hat{T} = (1/4)\left[\hat{z}(b_1^2\rho_2^2+b_2^2\rho_1^2+2b_1^2b_2^2\rho_1\rho_2) + (1+\hat{z}^2)b_1b_2\rho_1\rho_2(1-b_1b_2)\right]^{-1}$$
$$\times\left[2\hat{z}\left[b_1^2\rho_2^2(1+\rho_1^2) - b_1b_2\rho_1^2(1+\rho_2^2) + 2b_1^2b_2^2\rho_1\rho_2(1+\rho_1^2)\right]\right.$$

(continued)

$$- (1+\hat{z}^2)\rho_1\rho_2\left[b_1^2(1+\rho_2^2) - b_1b_2(1+\rho_1^2) + 2b_1^2b_2^2(1+\rho_1^2)\right]\Bigg] \quad . \tag{2.4.13}$$

Denote by $\hat{D}(z,b_1,b_2)$, the common denominator in (2.4.12) and (2.4.13). Then

$$\begin{aligned}\hat{D}(\hat{z},b_1,b_2) &= 4[\hat{z}(b_1^2\rho_2^2 + b_2^2\rho_1^2 + 2b_1^2b_2^2\rho_1\rho_2)+(1+\hat{z}^2)b_1b_2\rho_1\rho_2(1-b_1b_2)] \\ &= 4[\hat{z}(b_1\rho_2+b_2\rho_1)^2 - 2\hat{z}b_1b_2\rho_1\rho_2(1-b_1b_2) + (1+\hat{z}^2) \\ &\qquad \times b_1b_2\rho_1\rho_2(1-b_1b_2)] \\ &= 4[(1-\hat{z})^2b_1b_2\rho_1\rho_2(1-b_1b_2)+\hat{z}(b_1\rho_2+b_2\rho_1)^2]. \end{aligned} \tag{2.4.14}$$

With this notation, using (2.4.13), we have

$$\begin{aligned}\frac{1+\rho_1^2-2\hat{T}}{\rho_1} &= [D(\hat{z},b_1,b_2)\rho_1]^{-1}[4\hat{z}b_1^2\rho_2^2(1+\rho_1^2) + 4\hat{z}b_2^2\rho_1^2(1+\rho_1^2) \\ &+ 8\hat{z}b_1^2b_2^2\rho_1\rho_2(1+\rho_1^2) + 4(1+\hat{z}^2)b_1b_2\rho_1\rho_2(1+\rho_1^2) \\ &- 4(1+\hat{z}^2)b_1^2b_2^2\rho_1\rho_2(1+\rho_1^2) - 4\hat{z}b_1^2\rho_2^2(1+\rho_1^2) \\ &+ 4\hat{z}b_1b_2\rho_1^2(1+\rho_2^2) - 8b_1^2b_2^2\rho_1\rho_2(1+\rho_1^2) \\ &+ 2(1+\hat{z}^2)b_1^2\rho_1\rho_2(1+\rho_2^2) - 2(1+\hat{z}^2)b_1b_2\rho_1\rho_2(1+\rho_1^2) \\ &+ 4(1+\hat{z}^2)b_1^2b_2^2\rho_1\rho_2(1+\rho_1^2)] \quad ;\end{aligned}$$

that is

$$\begin{aligned}\frac{1+\rho_1^2-2\hat{T}}{\rho_1} &= 2[D(\hat{z},b_1,b_2)]^{-1}\{2\hat{z}[b_2^2\rho_1(1+\rho_1^2) + b_1b_2\rho_1(1+\rho_2^2)] \\ &\qquad + (1+\hat{z}^2)[b_1b_2\rho_2(1+\rho_1^2) + b_1^2\rho_2(1+\rho_2^2)]\}\end{aligned}$$

$$= \frac{2[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)][2\hat{z}b_2\rho_1 + (1+\hat{z}^2)b_1\rho_2]}{D(\hat{z},b_1,b_2)} . \qquad (2.4.15)$$

Similarly,

$$\frac{1+\rho_2^2-2\hat{S}}{\rho_2} = \frac{2[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)][2\hat{z}b_1\rho_2 + (1+\hat{z}^2)b_2\rho_1]}{D(\hat{z},b_1,b_2)} . \qquad (2.4.16)$$

Using (2.4.15) and (2.4.16),

$$\frac{\hat{z}}{(1+\hat{z})^2}\left[\frac{1+\rho_1^2-2\hat{T}}{\rho_1} + \frac{1+\rho_2^2-2\hat{S}}{\rho_2}\right]^2$$

$$= \frac{4\hat{z}(1+\hat{z})^2[b_1\rho_2+b_2\rho_1]^2[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^2}{[D(\hat{z},b_1,b_2)]^2} , \qquad (2.4.17)$$

$$\frac{(1+\rho_1^2-2\hat{T})(1+\rho_2^2-2\hat{S})}{\rho_1\rho_2}$$

$$= 4[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^2 \; [D(\hat{z},b_1,b_2)]^{-2}$$

$$\times\{(1-\hat{z})^4 b_1 b_2 \rho_1 \rho_2 + 2\hat{z}(b_1\rho_2+b_2\rho_1)^2[(1+\hat{z})^2 -2\hat{z}]\} , \qquad (2.4.18)$$

and, after simplifying,

$$\frac{\hat{T}}{b_1} + \frac{\hat{S}}{b_2}$$

$$= \frac{-2b_1b_2\rho_1\rho_2(1-\hat{z})^2[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]}{D(\hat{z},b_1,b_2)} . \qquad (2.4.19)$$

Recalling that $\hat{z} = z(\hat{T},\hat{S}) = a_2(\hat{T},\hat{S},0)$, the first equation of (2.4.7) may be written as

$$\frac{1+\hat{z}^2}{\hat{z}} - 2 = \frac{(1+\rho_1^2-2\hat{T})(1+\rho_2^2-2\hat{S})}{\rho_1\rho_2} - \frac{1}{\rho_1\rho_2}\left[\frac{\hat{T}}{b_1} + \frac{\hat{S}}{b_2}\right]^2$$

$$- \frac{\hat{z}}{(1+\hat{z})^2}\left[\frac{1+\rho_1^2-2\hat{T}}{\rho_1} + \frac{1+\rho_2^2-2\hat{S}}{\rho_2}\right]^2 .$$

Now, substitution of (2.4.17), (2.4.18), and (2.4.19) in the last expression gives

$$\frac{1+\hat{z}^2}{\hat{z}} - 2 = \frac{4[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^2}{[D(\hat{z},b_1,b_2)]^2}$$

$$\times[(1-\hat{z})^4 b_1b_2\rho_1\rho_2 + 2\hat{z}(1+\hat{z})^2(b_1\rho_2+b_2\rho_1)^2$$

$$- 4\hat{z}^2(b_1\rho_2+b_2\rho_1)^2 - (1-\hat{z})^4 b_1^2b_2^2\rho_1\rho_2$$

$$- \hat{z}(1+\hat{z})^2(b_1\rho_2+b_2\rho_1)^2]$$

$$= \frac{4[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^2}{[D(\hat{z},b_1,b_2)]^2}$$

$$\times[(1-\hat{z})^4 b_1b_2\rho_1\rho_2(1-b_1b_2) + \hat{z}(b_1\rho_2+b_2\rho_1)^2(1-\hat{z})^2]$$

$$= \frac{4[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^2}{[D(\hat{z},b_1,b_2)]^2}(1-\hat{z})^2$$

$$\times[(1-\hat{z})^2 b_1b_2\rho_1\rho_2(1-b_1b_2) + \hat{z}(b_1\rho_2+ b_2\rho_1)^2],$$

and hence, using (2.4.14),

$$\frac{1+\hat{z}^2-2\hat{z}}{\hat{z}} = \frac{[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)](1-\hat{z})^2}{D(\hat{z},b_1,b_2)} .$$

Thus,

$$\frac{D(\hat{z},b_1,b_2)}{\hat{z}} = [b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^2 . \qquad (2.4.20)$$

Again using (2.4.14), this is

$$4(1-\hat{z})^2 b_1 b_2 \rho_1 \rho_2 (1-b_1 b_2) + 4\hat{z}(b_1\rho_2+b_2\rho_1)^2$$

$$= \hat{z}[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^2 .$$

Putting the above equation into quadratic form, one obtains

$$4\hat{z}^2 b_1 b_2 \rho_1 \rho_2 (1-b_1 b_2) + \hat{z}\{4(b_1\rho_2+b_2\rho_1)^2 - 8b_1 b_2 \rho_1 \rho_2 (1-b_1 b_2)$$

$$-[b_1(1+\rho_2^2)+b_2(1+\rho_1^2)]^2\} + 4b_1 b_2 \rho_1 \rho_2 (1-b_1 b_2) = 0 ,$$

and, rearranging the coefficient of $\hat{z}$, one gets

$$4\hat{z}^2 b_1 b_2 \rho_1 \rho_2 (1-b_1 b_2) - \hat{z}\{[b_1(1-\rho_2)^2 + b_2(1-\rho_1)^2]$$

$$[b_1(1+\rho_2)^2 + b_2(1+\rho_1)^2] + 8b_1 b_2 \rho_1 \rho_2 (1-b_1 b_2)\}$$

$$+ 4b_1 b_2 \rho_1 \rho_2 (1-b_1 b_2) = 0 .$$

Finally, the value of $\hat{z}$ is one of

$$\hat{z} = [8b_1 b_2 \rho_1 \rho_2 (1-b_1 b_2)]^{-1}\{[b_1(1-\rho_2)^2 + b_2(1-\rho_1)^2]$$

$$[b_1(1+\rho_2)^2 + b_2(1+\rho_1)^2] + 8b_1 b_2 \rho_1 \rho_2 (1-b_1 b_2)$$

(continued)

$$\pm \Big[[b_1(1-\rho_2)^2+b_2(1-\rho_1)^2][b_1(1+\rho_2)^2+b_2(1+\rho_1)^2]$$

$$+8b_1b_2\rho_1\rho_2(1-b_1b_2)]^2 - 64b_1^2b_2^2\rho_1^2\rho_2^2(1-b_1b_2)^2\Big]^{1/2}$$

$$= 1 + \frac{[b_1(1-\rho_2)^2+b_2(1-\rho_1)^2][b_1(1+\rho_2)^2+ b_2(1+\rho_1)]}{8b_1b_2\rho_1\rho_2(1-b_1b_2)}$$

$$\pm \Bigg[\frac{[b_1(1-\rho_2)^2+b_2(1-\rho_1)^2]^2[b_1(1+\rho_2)^2+b_2(1+\rho_1)^2]^2}{64b_1^2b_2^2\rho_1^2\rho_2^2(1-b_1b_2)^2}$$

$$+ \frac{2[b_1(1-\rho_2)^2+b_2(1-\rho_1)^2][b_1(1+\rho_2)^2+b_2(1+\rho_1)^2]}{8b_1b_2\rho_1\rho_2(1-b_1b_2)}\Bigg]^{1/2},$$

which may be written as $\hat{z} = 1+V^2 \pm (V^2+2V)^{1/2}$, (2.4.21)

$$\text{where } V = \frac{[b_1(1-\rho_2)^2+b_2(1-\rho_1)^2][b_1(1+\rho_2)^2+b_2(1+\rho_1)^2]}{8b_1b_2\rho_1\rho_2(1-b_1b_2)}. \quad (2.4.22)$$

(The choice of sign is most easily discussed in the next section, and is postponed until then.)

Using the relation (2.4.20) for $D(\hat{z},b_1,b_2)$, we may further reduce (2.4.17), (2.4.18), and (2.4.19) to

$$\left.\begin{aligned}
\frac{1+\rho_1^2-2\hat{T}}{\rho_1} &= \frac{2[(1-\hat{z})^2b_1\rho_2+2\hat{z}(b_1\rho_2+b_2\rho_1)]}{\hat{z}[b_1(1+\rho_2^2)+b_2(1+\rho_1^2)]}, \\
\frac{1+\rho_2^2-2\hat{S}}{\rho_2} &= \frac{2[(1-\hat{z})^2b_2\rho_1+2\hat{z}(b_2\rho_1+b_1\rho_2)]}{\hat{z}[b_1(1+\rho_2^2)+b_2(1+\rho_1^2)]} \\
&\text{and} \\
\frac{\hat{T}}{b_1}+\frac{\hat{S}}{b_2} &= \frac{-2b_1b_2\rho_1\rho_2(1-\hat{z})^2}{\hat{z}[b_1(1+\rho_2^2)+b_2(1+\rho_1^2)]}.
\end{aligned}\right\} \quad (2.4.23)$$

$$\text{Thus } \hat{\zeta} = \frac{-\hat{z}}{1+\hat{z}}\left[\frac{1+\rho_1^2-2\hat{T}}{\rho_1} + \frac{1+\rho_2^2-2\hat{S}}{\rho_2}\right]$$

$$= \frac{-2[(1-\hat{z})^2(b_1\rho_2+b_2\rho_1) + 4\hat{z}(b_1\rho_2+b_2\rho_1)]}{(1+\hat{z})[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]}$$

$$= \frac{-2(1+\hat{z})(b_1\rho_2+b_2\rho_1)}{b_1(1+\rho_2^2) + b_2(1+\rho_1^2)} \quad . \tag{2.4.24}$$

To approximate the integral in (2.4.5), we also need expressions for $\hat{\beta}_{11}$, $\hat{\beta}_{12}$, and $\hat{\beta}_{21}$. Substituting equation (2.4.4) for U, with v = 0 in equations (2.4.3), we obtain

$$\hat{\beta}_{11} = \frac{(1-2\hat{T})(1-2\hat{S})}{\rho_1\rho_2} + 1 - \frac{1}{\rho_1\rho_2}\left[\frac{\hat{T}}{b_1} + \frac{\hat{S}}{b_2}\right]^2$$

$$= \frac{(1+\rho_1^2-2\hat{T})(1+\rho_2^2-2\hat{S})}{\rho_1\rho_2} + 2 - \frac{1}{\rho_1\rho_2}\left[\frac{\hat{T}}{b_1} + \frac{\hat{S}}{b_2}\right]^2$$

$$- [1 + (\rho_1/\rho_2)(1+\rho_2^2-2\hat{S}) + (\rho_2/\rho_1)(1+\rho_1^2-2\hat{T}) - \rho_1\rho_2]$$

$$= \frac{1+\hat{\zeta}^2+\hat{z}^2}{\hat{z}^2} - \left[1 + \frac{\rho_1}{\rho_2}(1+\rho_2^2-2\hat{S}) + \frac{\rho_2}{\rho_1}(1+\rho_1^2-2\hat{T}) - \rho_1\rho_2\right] \quad ,$$

$$\hat{\beta}_{12} = -(1-2\hat{T})/\rho_1 - (1+\rho_2^2-2\hat{S})/\rho_2$$

$$= -(1+\rho_1^2-2\hat{T})/\rho_1 - (1+\rho_2^2-2\hat{S})/\rho_2 + \rho_1$$

$$= \frac{(1+\hat{z})}{\hat{z}} + \rho_1 \quad ,$$

$$\text{and } \hat{\beta}_{21} = \frac{(1+\hat{z})}{\hat{z}} + \rho_2 \quad .$$

Hence,

$$\hat{z}(\hat{\beta}_{11}-\hat{z}) - \hat{z}^2(\hat{\beta}_{12}-\hat{\zeta})(\hat{\beta}_{21}-\hat{\zeta})$$

$$= (1+\hat{\zeta})^2 - \hat{z}[1+(\rho_1/\rho_2)(1+\rho_2^2-2\hat{S})+(\rho_2/\rho_1)(1+\rho_1^2-2\hat{T})-\rho_1\rho_2]$$

$$- (\hat{\zeta}+\hat{z}\rho_1)(\hat{\zeta}+\hat{z}\rho_2)$$

$$= 1 + \hat{\zeta}^2 - \hat{z}(1-\rho_1\rho_2) - \hat{\zeta}^2 - \hat{\zeta}\hat{z}(\rho_1+\rho_2) - \hat{z}^2\rho_1\rho_2$$

$$- \hat{z}[(\rho_1/\rho_2)(1+\rho_2^2-2\hat{S})+(\rho_2/\rho_1)(1+\rho_1^2-2\hat{T})]$$

$$= (1-\hat{z})(1+\hat{z}\rho_1\rho_2) - \hat{z}[(\rho_1/\rho_2)(1+\rho_2^2-2\hat{S})+(\rho_2/\rho_1)(1+\rho_1^2-2\hat{T})+\hat{\zeta}(\rho_1-\rho_2)],$$

and, finally, substituting the relations, (2.4.23) and (2.4.24), we obtain

$$\hat{z}(\hat{\beta}_{11}-\hat{z}) - \hat{z}^2(\hat{\beta}_{12}-\hat{\zeta})(\hat{\beta}_{21}-\hat{\zeta})$$

$$= (1-\hat{z})(1+\hat{z}\rho_1\rho_2) - 2[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^{-1}$$

$$\times [-\hat{z}(1+\hat{z})(b_1\rho_2+b_2\rho_1)\rho_1 - \hat{z}(1+\hat{z})(b_1\rho_2+b_2\rho_1)\rho_2$$

$$+ (1-\hat{z})^2 b_2\rho_1^2 + 2\hat{z}(b_1\rho_2+b_2\rho_1)\rho_1 + (1-\hat{z})^2 b_1\rho_2^2 + 2\hat{z}(b_1\rho_2+b_2\rho_1)\rho_2]$$

$$= (1-\hat{z})(1+\hat{z}\rho_1\rho_2) - 2[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^{-1}$$

$$\times [\hat{z}(1-\hat{z})(b_1\rho_2+b_2\rho_1)\rho_1 + \hat{z}(1-\hat{z})(b_1\rho_2+b_2\rho_1)\rho_2$$

$$+ (1-\hat{z})^2(b_1\rho_2^2+b_2\rho_1^2)]$$

$$= (1-\hat{z})[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^{-1} \times [b_1(1+\rho_2^2) + b_2(1+\rho_1^2)$$

$$+ \hat{z}b_1\rho_1\rho_2(1+\rho_2^2) + \hat{z}b_2\rho_1\rho_2(1+\rho_1^2) - 2\hat{z}(b_1\rho_2+b_2\rho_1)\rho_1$$

$$- 2\hat{z}(b_1\rho_2+b_2\rho_1)\rho_2 - 2(b_1\rho_2^2+b_2\rho_1^2) + 2\hat{z}(b_1\rho_2^2+b_2\rho_1^2)]$$

$$= (1-\hat{z})[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^{-1} \times [b_1(1-\rho_2^2) + b_2(1-\rho_1^2)$$

$$- \hat{z}\rho_1\rho_2(b_1+b_2-b_1\rho_2^2-b_2\rho_1^2)]$$

$$= \frac{(1-\hat{z})(1-\hat{z}\rho_1\rho_2)[b_1(1-\rho_2^2) + b_2(1-\rho_1^2)]}{b_1(1+\rho_2^2) + b_2(1+\rho_1^2)} . \tag{2.4.25}$$

Again, using relations, (2.4.23) and (2.4.24), we obtain

$$\left.\begin{aligned} (1-\hat{\zeta}+\hat{z}) &= 1 + \hat{z} + \frac{2(1+\hat{z})(b_1\rho_2+b_2\rho_1)}{b_1(1+\rho_2^2) + b_2(1+\rho_1^2)} \\ &= \frac{(1+\hat{z})[b_1(1+\rho_2)^2 + b_2(1+\rho_1)^2]}{b_1(1+\rho_2^2) + b_2(1+\rho_1^2)} , \\ \text{and} \quad (1+\hat{\zeta}+\hat{z}) &= \frac{(1+\hat{z})[b_1(1-\rho_2)^2 + b_2(1-\rho_1)^2]}{b_1(1+\rho_2^2) + b_2(1+\rho_1^2)} \end{aligned}\right\} \tag{2.4.26}$$

Then

$$(1-\hat{\zeta}+\hat{z})(1+\hat{\zeta}+\hat{z})$$

$$= \frac{(1+\hat{z})^2}{[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^2} \times \{b_1^2(1-\rho_2^2)^2 + b_2^2(1-\rho_1^2)^2$$

$$+ b_1b_2[(1+\rho_2)^2(1-\rho_1)^2 + (1+\rho_1)^2(1-\rho_2)^2]\}$$

$$= \frac{(1+\hat{z})^2\{b_1^2(1-\rho_2^2)^2 + b_2^2(1-\rho_1^2)^2 + 2b_1b_2[(\rho_1-\rho_2)^2 + (1-\rho_1\rho_2)^2]\}}{[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^2} ,$$

and thus

$$(1-\hat{\zeta}+\hat{z})^{1/2}(1+\hat{\zeta}+\hat{z})^{1/2}(1-\hat{z})$$

$$= [b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^{-1}(1+\hat{z})(1-\hat{z})$$

$$\times \{b_1^2(1-\rho_2^2)^2 + b_2^2(1-\rho_1^2)^2 + 2b_1b_2[(\rho_1-\rho_2)^2 + (1-\rho_1\rho_2)^2]\}^{1/2} . \tag{2.4.27}$$

Finally, using (2.4.25) and (2.4.27),

$$\frac{(1-\hat{\zeta}+\hat{z})^{1/2}(1+\hat{\zeta}+\hat{z})^{1/2}(1-\hat{z})}{\hat{z}(\hat{\beta}_{11}-\hat{z}) - \hat{z}^2(\hat{\beta}_{12}-\hat{\zeta})(\hat{\beta}_{21}-\hat{\zeta})}$$

$$= (1-\hat{z}\rho_1\rho_2)^{-1}[b_1(1-\rho_2^2) + b_2(1-\rho_1^2)]^{-1}(1+\hat{z})$$

$$\times\{b_1^2(1-\rho_2^2)^2 + b_2^2(1-\rho_1^2)^2 + 2b_1b_2[(\rho_1-\rho_2)^2 + (1-\rho_1\rho_2)^2]\}^{1/2} . \tag{2.4.28}$$

It remains only to calculate $\left(\frac{\partial \hat{a}_2}{\partial v}\right)^2\Big|_{v=0}$, and

$$\left[\frac{\partial^2 z}{\partial T^2}(\hat{T},\hat{S})\right]\left[\frac{\partial^2 z}{\partial S^2}(\hat{T},\hat{S})\right] - \left[\frac{\partial^2 z}{\partial T \partial S}(\hat{T},\hat{S})\right]^2 .$$

To calculate the first quantity, use (2.4.2), with the substitution for U given in (2.4.4); that is,

$$a_2 + \frac{1}{a_2} + \frac{a_2}{(1+a_2)^2}\left[\frac{1+\rho_1^2-2T}{\rho_1} + \frac{1+\rho_2^2-2S}{\rho_2}\right]^2$$

$$= 2 + \frac{(1+\rho_1^2-2T)(1+\rho_2^2-2S)}{\rho_1\rho_2} - \frac{1}{\rho_1\rho_2}\left[\frac{v(b_1+b_2)}{b_1b_2} - \frac{T}{b_1} - \frac{S}{b_2}\right]^2 . \tag{2.4.29}$$

Differentiating with respect to v, we get

$$\left(1 - \frac{1}{a_2}\right) \frac{\partial a_2}{\partial v} + \left[\frac{1+\rho_1^2-2T}{\rho_1} + \frac{1+\rho_2^2-2S}{\rho_2}\right]^2 \frac{\partial a_2}{\partial v}$$

$$= \frac{-2(b_1+b_2)}{\rho_1\rho_2 b_1 b_2}\left[\frac{v(b_1+b_2)}{b_1 b_2} - \frac{T}{b_1} - \frac{S}{b_2}\right] .$$

Then, setting $v=0$, $T=\hat{T}$, $S=\hat{S}$, and, hence, $a_2=\hat{z}$, and solving for the derivative, we obtain

$$\left.\frac{\partial a_2}{\partial v}\right|_{v=0,T=\hat{T},S=\hat{S}}$$

$$= \frac{2(b_1+b_2)(b_1b_2\rho_1\rho_2)^{-1}(\hat{T}/b_1 + \hat{S}/b_2)}{(1-\hat{z}^2)\hat{z}^{-2} + (1-\hat{z})(1+\hat{z})^{-3}[(1+\rho_1^2-2\hat{T})/\rho_1 + (1+\rho_2^2-2\hat{S})/\rho_2]^2} ,$$

and, substituting for the expressions containing $\hat{T}$ and $\hat{S}$ using (2.4.23),

$$\left.\frac{\partial a_2}{\partial v}\right|_{v=0,T=\hat{T},S=\hat{S}} = \frac{2(b_1+b_2)\left[\dfrac{2(1-\hat{z})^2}{\hat{z}[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]}\right]}{\dfrac{1-\hat{z}^2}{\hat{z}^2} - \dfrac{1-\hat{z}}{(1+\hat{z})^3}\left[\dfrac{4(1+\hat{z})^4(b_1\rho_2+b_2\rho_1)^2}{\hat{z}^2[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^2}\right]}$$

$$= \frac{\dfrac{4(b_1+b_2)(1-\hat{z})^2}{\hat{z}[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]}}{\dfrac{1-\hat{z}^2}{\hat{z}}\left[1 - \dfrac{4(b_1\rho_2+b_2\rho_1)}{[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^2}\right]}$$

$$= \frac{4(b_1+b_2)(1-\hat{z})^2\hat{z}[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]}{(1-\hat{z})(1+\hat{z})\{[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^2 - 4(b_1\rho_2+b_2\rho_1)\}}$$

$$= \frac{4\hat{z}(1-\hat{z})(b_1+b_2)[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]}{(1+\hat{z})[b_1(1-\rho_2)^2 + b_2(1-\rho_1)^2][b_1(1+\rho_2)^2 + b_2(1+\rho_1)^2]}$$

$$= \frac{4\hat{z}(1-\hat{z})(b_1+b_2)[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]}{(1+\hat{z})\{b_1^2(1-\rho_2^2)^2 + b_2^2(1-\rho_1^2)^2 + 2b_1b_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\}} .$$

Thus, combining the above with equation (2.4.28), $\phi(\hat{T},\hat{S})$ in equation (2.4.6) can finally be expressed as

$$\frac{(1+\hat{\zeta}+\hat{z})^{1/2}(1-\hat{\zeta}+\hat{z})^{1/2}(1-\hat{z})}{\hat{z}(\hat{\beta}_{11}-\hat{z}) - \hat{z}^2(\hat{\beta}_{12}-\hat{\zeta})(\hat{\beta}_{21}-\hat{\zeta})} \left(\frac{\partial a_2}{\partial v}\right)^2 \Bigg|_{v=0,T=\hat{T},S=\hat{S}}$$

$$= 16\hat{z}^2(1-\hat{z})^2(b_1+b_2)^2[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^2$$

$$\times \{(1+\hat{z})(1-\hat{z}\rho_1\rho_2)[b_1(1-\rho_2^2) + b_2(1-\rho_1^2)]\}^{-1}$$

$$\times\{b_1^2(1-\rho_2^2)^2 + b_2^2(1-\rho_1^2)^2 + 2b_1b_2[(\rho_1-\rho_2)^2 + (1-\rho_1\rho_2)^2]\}^{-3/2} . \tag{2.4.30}$$

To calculate the remaining derivatives, (2.4.29) is again used. v is set equal to 0, and the resulting equation is differentiated with respect to T and with respect to S.

$$\text{Thus, } (1-1/z^2)\frac{\partial z}{\partial T} + \left[\frac{1+\rho_1^2-2T}{\rho_1} + \frac{1+\rho_2^2-2S}{\rho_2}\right]^2 \frac{1-z}{(1+z)^3}\frac{\partial z}{\partial T}$$

$$- \frac{4z}{(1+z)^2\rho_1}\left[\frac{1+\rho_1^2-2T}{\rho_1} + \frac{1+\rho_2^2-2S}{\rho_2}\right]$$

(continued)

$$= \frac{-2(1+\rho_2^2-2S)}{\rho_1\rho_2} - \frac{2}{b_2\rho_1\rho_2}\left[\frac{T}{b_1}+\frac{S}{b_2}\right] , \qquad (2.4.31)$$

and $$(1-1/z^2)\frac{\partial z}{\partial S} + \left[\frac{1+\rho_1^2-2T}{\rho_1}+\frac{1+\rho_2^2-2S}{\rho}\right]^2 \frac{1-z}{(1+z)^3}\frac{\partial z}{\partial S}$$

$$- \frac{4z}{(1+z)^2\rho_2}\left[\frac{1+\rho_1^2-2T}{\rho_1}+\frac{1+\rho_2^2-2S}{\rho_2}\right]$$

$$= \frac{-2(1+\rho_1^2-2T)}{\rho_1\rho_2} - \frac{2}{b_2\rho_1\rho_2}\left[\frac{T}{b_1}+\frac{S}{b_2}\right] . \qquad (2.4.32)$$

If (2.4.31) is differentiated with respect to T, and then evaluated at the saddle-point, $(\hat{T},\hat{S})$, where first order derivatives vanish, the following equation is obtained:

$$(1-1/\hat{z}^2)\left.\frac{\partial^2 z}{\partial T^2}\right|_{(\hat{T},\hat{S})} + \left[\frac{1+\rho_1^2-2\hat{T}}{\rho_1}+\frac{1+\rho_2^2-2\hat{S}}{\rho_2}\right]^2 \frac{1-\hat{z}}{(1+\hat{z})^3}\left.\frac{\partial^2 z}{\partial T^2}\right|_{(\hat{T},\hat{S})}$$

$$+ \frac{8\hat{z}}{(1+\hat{z})^2\rho_1^2} = \frac{-2}{b_1^2\rho_1\rho_2} ,$$

and hence,

$$\left.\frac{\partial^2 z}{\partial T^2}\right|_{(\hat{T},\hat{S})} = \frac{\dfrac{2}{b_1^2\rho_1\rho_2} + \dfrac{8\hat{z}}{(1+\hat{z})^2\rho_2^2}}{\dfrac{1-\hat{z}^2}{\hat{z}^2} - \dfrac{1-\hat{z}}{(1+\hat{z})^3}\left[\dfrac{1+\rho_1^2-2\hat{T}}{\rho_1}+\dfrac{1+\rho_2^2-2\hat{S}}{\rho_2}\right]^2}$$

$$= \frac{\left(\dfrac{2}{\rho_1}\right)\left[\dfrac{(1+\hat{z})^2\rho_1 + 4\hat{z}b_1^2\rho_2}{(1+\hat{z})^2 b_1^2\rho_1\rho_2}\right]}{\dfrac{(1-\hat{z}^2)}{\hat{z}^2}\left[1 - \dfrac{4(b_1\rho_2+b_2\rho_1)^2}{[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^2}\right]}$$

$$= 2\hat{z}^2[(1+\hat{z})^2\rho_1 + 4\hat{z}b_1^2\rho_2][b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^2$$

$$\times \Big[\rho_1(1-\hat{z})(1+\hat{z})^3 b_1^2\rho_1\rho_2\{b_1^2(1-\rho_2^2)^2 + b_2^2(1-\rho_1^2)^2$$

$$+2b_1b_2[(\rho_1-\rho_2)^2 + (1-\rho_1\rho_2)^2]\}\Big]^{-1} \quad . \tag{2.4.33}$$

Similarly,

$$\left.\frac{\partial^2 z}{\partial S^2}\right|_{(\hat{T},\hat{S})} = 2\hat{z}^2[(1+\hat{z})^2\rho_2 + 4\hat{z}b_2^2\rho_1][b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^2$$

$$\times \Big[\rho_2(1-\hat{z})(1+\hat{z})^3 b_2^2\rho_1\rho_2\{b_1^2(1-\rho_2^2)^2 + b_2^2(1-\rho_1^2)^2$$

$$+2b_1b_2[(\rho_1-\rho_2)^2 + (1-\rho_1\rho_2)^2]\}\Big]^{-1} \quad , \tag{2.4.34}$$

and

$$(1-1/\hat{z}^2)\left.\frac{\partial^2 z}{\partial T\partial S}\right|_{(\hat{T},\hat{S})} + \left[\frac{1+\rho_1^2-2\hat{T}}{\rho_1} + \frac{1+\rho_2^2-2\hat{S}}{\rho_2}\right]^2 \frac{1-\hat{z}}{(1+\hat{z})^3}\left.\frac{\partial^2 z}{\partial T\partial S}\right|_{(\hat{T},\hat{S})}$$

$$+ \frac{8\hat{z}}{(1+\hat{z})^2\rho_1\rho_2} = \frac{4}{\rho_1\rho_2} - \frac{2}{b_1b_2\rho_1\rho_2} \quad ,$$

so that

$$\left.\frac{\partial^2 z}{\partial T \partial S}\right|_{(\hat{T},\hat{S})} = \frac{\dfrac{8\hat{z}}{(1+\hat{z})^2\rho_1\rho_2} - \dfrac{4}{\rho_1\rho_2} + \dfrac{2}{b_1 b_2 \rho_1 \rho_2}}{\dfrac{1-\hat{z}^2}{\hat{z}^2} - \dfrac{1-\hat{z}}{(1+\hat{z})^3}\left[\dfrac{1+\rho_1^2-2\hat{T}}{\rho_1} + \dfrac{1+\rho_2^2-2\hat{S}}{\rho_2}\right]^2}$$

$$= \frac{\dfrac{2}{\rho_1\rho_2} \cdot \dfrac{4\hat{z}b_1b_2 - 2(1+\hat{z})^2 b_1 b_2 + (1+\hat{z})^2}{(1+\hat{z})^2 b_1 b_2}}{\dfrac{1-\hat{z}^2}{\hat{z}^2}\left[1 - \dfrac{4(b_1\rho_2+b_2\rho_1)^2}{b_1(1+\rho_2^2) + b_2(1+\rho_1^2)}\right]}$$

$$= 2\hat{z}^2[4\hat{z}b_1b_2 - 2(1+\hat{z})^2 b_1 b_2 + (1+\hat{z})^2]$$

$$\times [b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^2 [b_1 b_2 \rho_1 \rho_2 (1-\hat{z})(1+\hat{z})^3]^{-1}$$

$$\times \left[b_1^2(1-\rho_2^2)^2 + b_2^2(1+\rho_1^2)^2 + 2b_1b_2[(\rho_1-\rho_2)^2 + (1-\rho_1\rho_2)]^{-1}\right] . \tag{2.4.35}$$

Thus, from (2.4.33), (2.4.34), and (2.4.35),

$$\left[\left[\frac{\partial^2 z}{\partial T^2}(\hat{T},\hat{S})\right]\left[\frac{\partial^2 z}{\partial S^2}(\hat{T},\hat{S})\right] - \left[\frac{\partial^2 z}{\partial T \partial S}(\hat{T},\hat{S})\right]^2\right]^{1/2}$$

$$= 2\hat{z}^2[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^2[b_1b_2(\rho_1\rho_2)^{3/2}(1-\hat{z})(1+\hat{z})^3]^{-1}$$

$$\times \{b_1^2(1-\rho_2^2)^2 + b_2^2(1-\rho_1^2)^2 + 2b_1b_2[(\rho_1-\rho_2)^2 + (1-\rho_1\rho_2)^2]\}^{-1}$$

$$\times \{[(1+\hat{z})^2\rho_1 + 4\hat{z}b_1^2\rho_2][(1+\hat{z})^2\rho_2 + 4b_2^2\rho_1]$$

$$- \rho_1\rho_2[4\hat{z}b_1b_2 - 2(1+\hat{z})^2 b_1 b_2 + (1+\hat{z})^2]^2\}^{1/2} .$$

Simplifying the numerator, we see that

$$\{[(1+\hat{z})^2\rho_1 + 4\hat{z}b_1^2\rho_2][(1+\hat{z})^2\rho_2 + 4\hat{z}b_2^2\rho_1]$$

$$-\rho_1\rho_2[4\hat{z}b_1b_2 - 2(1+\hat{z})^2b_1b_2 + (1+\hat{z})^2]^2\}^{1/2}$$

$$= \{(1+\hat{z})^2\rho_1\rho_2 + 16\hat{z}^2b_1^2b_2^2\rho_1\rho_2 + 4\hat{z}(1+\hat{z})^2(b_1^2\rho_2^2+b_2^2\rho_1^2)$$

$$- 16\hat{z}^2b_1^2b_2^2\rho_1\rho_2 - 4(1+\hat{z})^4b_1^2b_2^2\rho_1\rho_2 - (1+\hat{z})^4\rho_1\rho_2$$

$$+ 16\hat{z}^2(1+\hat{z})^2b_1^2b_2^2\rho_1\rho_2 - 8\hat{z}(1+\hat{z})^2b_1b_2\rho_1\rho_2 + 4(1+\hat{z})^4b_1b_2\rho_1\rho_2\}^{1/2}$$

$$= 2(1+\hat{z})[\hat{z}(b_1^2\rho_2^2+b_2^2\rho_1^2) + (1+\hat{z}^2+2\hat{z})b_1^2b_2^2\rho_1\rho_2 + 4\hat{z}b_1^2b_2^2\rho_1\rho_2$$

$$- 2\hat{z}b_1b_2\rho_1\rho_2 + (1+\hat{z}^2+2\hat{z})b_1b_2\rho_1\rho_2]^{1/2}$$

$$= 2(1+\hat{z})[\hat{z}(b_1\rho_2+b_2\rho_1)^2 + b_1b_2\rho_1\rho_2(1-b_1b_2)(1+\hat{z})^2]^{1/2} \quad .$$

Hence, using the definition, and subsequent expression, for $D(z,b_1,b_2)$, equations (2.4.14) and (2.4.20) respectively, the numerator is

$$\hat{z}^{1/2}(1+\hat{z})[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)] \quad .$$

Then $$\left[\left[\frac{\partial^2 z}{\partial T^2}(\hat{T},\hat{S})\right]\left[\frac{\partial^2 z}{\partial S^2}(\hat{T},\hat{S})\right] - \left[\frac{\partial^2 z}{\partial T\partial S}(\hat{T},\hat{S})\right]^2\right]^{1/2}$$

$$= 2\hat{z}^{5/2}[b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^3[b_1b_2(\rho_1\rho_2)^{3/2}(1-\hat{z})(1+\hat{z})^2]^{-1}$$

$$\times\{b_1^2(1-\rho_2^2)^2 + b_2^2(1-\rho_1^2)^2 + 2b_1b_2[(\rho_1-\rho_2)^2 + (1-\rho_1\rho_2)^2]\}^{-1} \quad (2.4.36)$$

Finally, from equations (2.4.30) and (2.4.36), the bivariate saddle-point approximation of the joint marginal density function of b_1 and b_2 for known means is

$$h(b_1,b_2) \sim \frac{(1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}(n/2)(n-2)}{2(\rho_1\rho_2)^{n/2}(b_1+b_2)^2}$$

$$\times\ (2\pi i)^{-2}\iint a_2^{n/2-2}\ \frac{(1-a_1+a_2)^{1/2}(1+a_1+a_2)^{1/2}(1-a_2)}{a_2(\beta_{11}-a_2)\ -\ a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)}$$

$$\times \left(\frac{\partial a_2}{\partial v}\right)^2\Bigg|_{v=0}\ dSdT$$

$$\sim \frac{(1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}(n/2)(n-2)}{2(\rho_1\rho_2)^{n/2}(b_1+b_2)^2} \times \frac{\hat{z}^{(n-2)/2}}{2\pi(n/2-2)}$$

$$\times\ 8\hat{z}^{1/2}(1-\hat{z})^3(1+\hat{z})(\rho_1\rho_2)^{3/2}b_1b_2(b_1+b_2)^2$$

$$\times\ \{(1-\hat{z}\rho_1\rho_2)[b_1(1+\rho_2^2)\ +\ b_2(1+\rho_1^2)][b_1(1-\rho_2^2)\ +\ b_2(1-\rho_1^2)]\}^{-1}$$

$$\times\ \{b_1^2(1-\rho_2^2)^2\ +\ b_2^2(1-\rho_1^2)^2\ +\ 2b_1b_2[(\rho_1-\rho_2)^2\ +\ (1-\rho_1\rho_2)^2]\}^{-1/2},$$

and after cancellation,

$$h(b_1,b_2) \sim \frac{2n(n-2)}{\pi(n-4)} \times \frac{(1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}(1-\hat{z})^3(1+\hat{z})b_1b_2}{(1-\hat{z}\rho_1\rho_2)}$$

$$\times \left(\frac{\hat{z}}{\rho_1\rho_2}\right)^{(n-3)/2}\ [b_1(1+\rho_2^2)\ +\ b_2(1+\rho_1^2)]^{-1}$$

$$\times\ [b_1(1-\rho_2^2)\ +\ b_2(1-\rho_1^2)]^{-1} \times \{b_1^2(1-\rho_2^2)^2\ +\ b_2^2(1-\rho_1^2)^2$$

$$+\ 2b_1b_2[(\rho_1-\rho_2)^2\ +\ (1-\rho_1\rho_2)^2]\}^{-1/2}, \qquad (2.4.37)$$

where the error is relatively $O(n^{-1})$.

(b) Fitted Means

Comparing equations (2.4.1) and (2.2.53), written below,

$$M^*(T,S,U) \sim \frac{(1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}}{(1-\rho_1)(1-\rho_2)} \cdot \frac{a_2^{(n-1)/2}}{(\rho_1\rho_2)^{(n-1)/2}}$$

$$\times \frac{(1-a_1+a_2)^{1/2}(1+a_1+a_2)^{3/2}(1-a_2)}{a_2(\beta_{11}-a_2) - a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)}, \quad (2.4.38)$$

we see that $M^*(T,S,U)$ is obtained from $M(T,S,U)$ by making the following two changes:

(1) n is replaced by $(n-1)$,

and (2) $M(T,S,U)$ is multiplied by $\dfrac{1+a_1+a_2}{(1-\rho_1)(1-\rho_2)}$.

The first change will induce the corresponding replacement of n by $(n-1)$ in the approximation for $h(b_1,b_2)$. The second change will induce the corresponding multiplication of the approximation for $h(b_1,b_2)$ by $(1+\hat{\zeta}+\hat{z})/[(1-\rho_1)(1-\rho_2)]$, since only the factor corresponding to $\phi(\hat{z}_1,\hat{z}_2)$ in (2.4.6) is affected.

Therefore, using relation (2.4.37), for $h(b_1,b_2)$, and equation (2.4.26) for $(1+\hat{\zeta}+\hat{z})$, the joint density, $h^*(b_1^*,b_2^*)$, for the regression coefficients with fitted means is approximated as follows:

$$h^*(b_1^*, b_2^*) \sim \frac{2(n-1)(n-3)}{\pi(n-5)} \times \frac{(1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}(1-\hat{z})^3(1+\hat{z})^2 b_1 b_2}{(1-\rho_1)(1-\rho_2)(1-\hat{z}\rho_1\rho_2)}$$

$$\times [\hat{z}/\rho_1\rho_2]^{(n-4)/2} [b_1(1-\rho_2)^2 + b_2(1-\rho_1)^2]$$

$$\times [b_1(1-\rho_2^2) + b_2(1-\rho_1^2)]^{-1} [b_1(1+\rho_2^2) + b_2(1+\rho_1^2)]^{-2}$$

$$\times \{b_1^2(1-\rho_2^2)^2 + b_2^2(1-\rho_1^2)^2 + 2b_1b_2[(\rho_1-\rho_2)^2$$

$$+ (1-\rho_1\rho_2)]\}^{-1/2}, \quad (2.4.38)$$

where the relative error is $O(n^{-1})$.

2.5 The Approximate Marginal Densities for Regressing y on x

The marginal density of b_1 could have been obtained from $h(b_1,b_2)$ by integrating over b_2. It was found, however, to be more convenient to perform the following transformation first:

$$\omega = b_1 \ , \quad \xi = b_1/b_2 \ .$$

Then the density of $\omega = b_1$ is given by

$$f(\omega) = \int g(\omega,\xi)\,d\xi \ , \quad \text{where } g \text{ is the joint density of } \omega \text{ and } \xi. \tag{2.5.1}$$

But the Jacobian of the transformation is found from

$$\frac{\partial(b_1,b_2)}{\partial(\omega\ ,\xi\ )} = \begin{vmatrix} 1 & 0 \\ 1/\xi & -\omega/\xi^2 \end{vmatrix} = -\frac{\omega}{\xi^2} \ ,$$

so that $\left|\dfrac{\partial(b_1,b_2)}{\partial(\omega\ ,\xi\ )}\right| = \dfrac{|\omega|}{\xi^2}$.

Hence,
$$g(\omega,\xi) = f(\omega,\omega/\xi) \times \frac{|\omega|}{\xi^2}$$

$$\sim [\chi(\omega,\xi)]^{(n-3)/2}\,\theta(\omega,\xi), \tag{2.5.2}$$

where
$$\chi(\omega,\xi) = \hat{z}/\rho_1\rho_2 \ , \tag{2.5.3}$$

and
$$\theta(\omega,\xi) = \frac{2(1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}n(n-2)}{\pi(n-4)(1-\rho_1\rho_2\hat{z})[\omega(1-\rho_2^2) + (\omega/\xi)(1-\rho_1^2)]}$$

$$\times \frac{|\omega|}{\xi^2} \times \frac{(1-\hat{z})^2(1-\hat{z}^2)\,\omega(\omega/\xi)}{[\omega(1+\rho_2^2) + (\omega/\xi)(1+\rho_1^2)]}$$

$$\times \{\omega^2(1-\rho_2^2)^2 + (\omega^2/\xi^2)(1-\rho_1^2)^2 + 2(\omega^2/\xi)[(\rho_1-\rho_2)^2$$
$$+ (1-\rho_1\rho_2)^2]\}^{-1/2}$$

$$= \frac{2(1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}n(n-2)}{\pi(n-4)(1-\rho_1\rho_2\ \hat{z})}$$

$$\times \frac{(1-\hat{z})^2(1-\hat{z}^2)}{[\xi(1-\rho_2^2) + (1-\rho_1^2)][\xi(1+\rho_2^2) + (1+\rho_1^2)]}$$

$$\times \{\xi^2(1-\rho_2^2)^2 + (1-\rho_1^2)^2 + 2\xi[\rho_1-\rho_2)^2 + (1-\rho_1\rho_2)^2]\}^{1/2} \quad .$$

$$(2.5.4)$$

Observe that, using Schwartz's inequality,

$$\xi = \frac{b_1}{b_2} = \frac{\Sigma x_i y_i}{\Sigma x_i^2} \times \frac{\Sigma y_i^2}{\Sigma x_i y_i}$$

$$= \frac{\Sigma y_i^2}{\Sigma x_i^2}$$

$$= \frac{(\Sigma x_i^2)(\Sigma y_i^2)}{(\Sigma x_i^2)^2}$$

$$\geqq \frac{(\Sigma x_i y_i)^2}{(\Sigma x_i^2)^2}$$

$$= \omega^2$$

Hence, $f(\omega) \sim \int_{\omega^2}^{\infty} [\chi(\omega,\xi)]^{(n-3)/2}\ \theta(\omega,\xi)d\xi \quad .$ (2.5.5)

But $$V = \frac{[\omega(1-\rho_2)^2 + (\omega/\xi)(1-\rho_1)^2][\omega(1+\rho_2)^2 + (\omega/\xi)(1+\rho_1)^2]}{8\rho_1\rho_2(\omega^2/\xi)[1-(\omega^2/\xi)]}$$

$$= \frac{[\xi(1-\rho_2)^2 + (1-\rho_1)^2][\xi(1+\rho_2)^2 + (1+\rho_1)^2]}{8\rho_1\rho_2(\xi-\omega^2)}, \quad (2.5.6)$$

and the numerator of V is given by

$$\xi^2(1-\rho_2^2)^2 + \xi[(1-\rho_1)^2(1+\rho_2)^2 + (1+\rho_1)^2(1-\rho_2)^2] + (1-\rho_1^2)^2$$

$$= [\xi(1-\rho_2^2) + (1-\rho_1^2)]^2 + \xi[(1+\rho_1)(1-\rho_2) + (1-\rho_1)(1+\rho_2)]^2 > 0.$$

Therefore, if $\rho_1\rho_2 > 0$, for fixed ω, $V \to \infty$ as $\xi \to \infty$ or as $\xi \downarrow \omega^2$ (except in the extreme cases, $\rho_1 = \pm 1$ or $\rho_2 = \pm 1$). Similarly, if $\rho_1\rho_2 > 0$, for fixed ω, $V \to -\infty$ as $\xi \to \infty$ or as $\xi \downarrow \omega^2$.

Furthermore,

$$\frac{\partial V}{\partial \xi} = [8\rho_1\rho_2(\xi-\omega^2)^2]^{-1}\{(\xi-\omega^2)[(1-\rho_2)^2[\xi(1+\rho_2)^2+(1+\rho_1)^2]$$
$$+ (1+\rho_2)^2[\xi(1-\rho_2)^2+(1-\rho_1)^2]] - [\xi(1-\rho_2)^2+(1-\rho_1)^2]$$
$$\times [\xi(1+\rho_2)^2+(1+\rho_1)^2]\} , \tag{2.5.7}$$

and $\dfrac{\partial V}{\partial \xi} = 0$ iff

$$\xi^2(1-\rho_2^2)^2 - 2\omega^2\xi(1-\rho_2^2)^2 - \{\omega^2(1-\rho_2)^2(1+\rho_1)^2+\omega^2(1+\rho_2)^2(1-\rho_1)^2$$
$$+(1-\rho_1^2)^2\} = 0 .$$

Hence, $\dfrac{\partial V}{\partial \xi} = 0$ iff

$$\xi = [2(1-\rho_2^2)^2]^{-1}\{2\omega^2(1-\rho_2^2)^2 \pm [4\omega^4(1-\rho_2^2)^4+4\omega^2(1-\rho_2^2)^2(1-\rho_2)^2 \times (1+\rho_1)^2$$
$$+4\omega^2(1-\rho_2^2)^2(1+\rho_2)^2(1-\rho_1)^2+4(1-\rho_2^2)^2(1-\rho_1^2)^2]^{1/2}\}$$

$$= \omega^2 \pm \frac{2\{[(1-\rho_2)^2\omega^2+(1-\rho_1)^2][(1+\rho_2)^2\omega^2+(1+\rho_1)^2]\}^{1/2}}{2(1-\rho_2^2)^2}$$

But $\xi \geqq \omega^2$, and the only critical value of ξ is

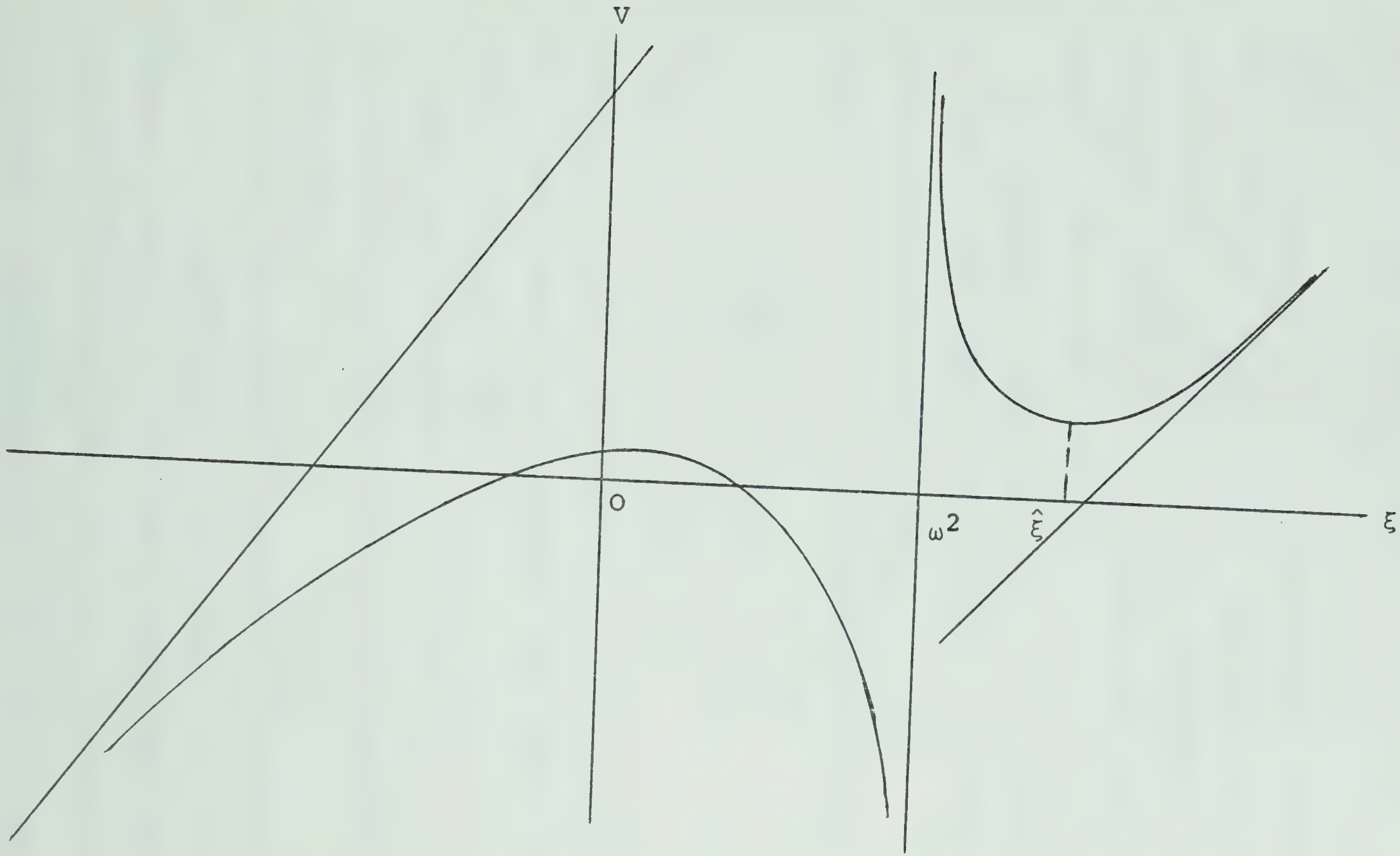


Figure 2: The Graph of $V(\xi,\omega)$ for Fixed ω.

$$\hat{\xi}(\omega) = \omega^2 + \left[\left[\omega^2 + \left(\frac{1-\rho_1}{1-\rho_2}\right)^2 \right] \left[\omega^2 + \left(\frac{1+\rho_1}{1+\rho_2}\right)^2 \right] \right]^{1/2} \qquad (2.5.8)$$

Since for $\rho_1\rho_2 > 0$, $V \to \infty$ at the extreme values of the range of ξ, $V(\hat{\xi}(\omega),\omega)$ is the minimum value of $V(\xi,\omega)$ for fixed ω, and since for $\rho_1\rho_2 < 0$, $V \to -\infty$ at the extreme values of the range of ξ, $V(\hat{\xi}(\omega),\omega)$ is the maximum value of $V(\xi,\omega)$ for fixed ω. Henceforth, $V(\hat{\xi}(\omega),\omega)$ will be abbreviated as $\hat{V}(\omega)$.

The ambiguity in equation (2.4.21) for $\hat{z}$ will now be resolved. Recall that $\hat{z} = a_2(\hat{T},\hat{S},0)$, by definition.
From equation (2.2.27), $a_2 = \phi_1\phi_2$,
where ϕ_1 and ϕ_2 are roots of the equation (2.2.8),

$$\phi^4 + x_1\phi^3 + x_2\phi^2 + x_1\phi + 1 = 0,$$

satisfying $|\phi_1| < 1$ and $|\phi_2| < 1$.

Relations (2.2.28) and (2.2.29), which were used to solve for $\hat{z}$, were derived using only the fact that ϕ_1 and ϕ_2 were roots of the above equation.

Using the restrictions, $|\phi_1| < 1$ and $|\phi_2| < 1$, it is clear that the correct value of $\hat{z}$ is that one which has smallest absolute value.
Note that $[\hat{V}^2(\omega)+2\hat{V}(\omega)]^{1/2}$ is real iff $\hat{V}^2(\omega)+2\hat{V}(\omega) \geq 0$;
i. e. iff $\hat{V}(\omega) \geq 0$ or $\hat{V}(\omega) \leq -2$.
However, for $\rho_1\rho_2 > 0$, $\hat{V}(\omega)$ has already been shown to be non-negative. It follows, then, for $\rho_1\rho_2 > 0$, since $1+\hat{V}(\omega) > 0$,

$$|1 + \hat{V}(\omega) - [\hat{V}^2(\omega)+2\hat{V}(\omega)]^{1/2}| < |1 + \hat{V}(\omega) + [\hat{V}^2(\omega)+2\hat{V}(\omega)]| \; .$$

Hence, $\hat{z} = 1+\hat{V}(\omega) - [\hat{V}^2(\omega)+2\hat{V}(\omega)]^{1/2}, \quad (\rho_1\rho_2>0) \quad (2.5.9)$

For $\rho_1\rho_2 < 0$, using (2.5.6) and (2.5.7) evaluated at $\xi = \hat{\xi}$,

$$\hat{V}(\omega) = \frac{[\hat{\xi}(1-\rho_2)^2+(1-\rho_1)^2][\hat{\xi}(1+\rho_2)^2+(1+\rho_1)^2]}{8\rho_1\rho_2(\hat{\xi}-\omega^2)}$$

$$= \frac{(1-\rho_2)^2[\hat{\xi}(1+\rho_2)^2+(1+\rho_1)^2] + (1+\rho_2)^2[\hat{\xi}(1-\rho_2)^2+(1-\rho_1)^2]}{8\rho_1\rho_2}$$

Substituting the expression (2.5.8) for $\hat{\xi}$,

$$\hat{V}(\omega) = [8\rho_1\rho_2]^{-1}\left[2(1-\rho_2^2)^2\left[\left[\omega^2 + \left(\frac{1-\rho_1}{1-\rho_2}\right)^2\right]\left[\omega^2 + \left(\frac{1+\rho_1}{1+\rho_2}\right)^2\right]\right]^{1/2}\right.$$

$$\left. +2(1-\rho_2^2)^2\omega^2+(1-\rho_2)^2(1+\rho_1)^2+(1+\rho_2)^2(1-\rho_1)^2\right]. \quad (2.5.10)$$

The numerator is minimized with respect to ω when $\omega = 0$, and the numerator is always positive.

Therefore, when $\rho_1\rho_2 < 0$,

$$\hat{V}(\omega) \leqq \hat{V}(0) = \frac{2(1-\rho_1^2)(1-\rho_2^2) + (1-\rho_2)^2(1+\rho_1)^2 + (1+\rho_2)^2(1-\rho_1)^2}{8\rho_1\rho_2}$$

$$= \frac{[(1-\rho_2)(1+\rho_1) + (1-\rho_1)(1+\rho_2)]^2}{8\rho_1\rho_2}$$

$$= \frac{4(1-\rho_1\rho_2)^2}{8\rho_1\rho_2}$$

$$= \frac{(1-\rho_1\rho_2)^2}{2\rho_1\rho_2} \quad .$$

It is easily seen that this last quantity is bounded above by -2, for $\rho_1\rho_2$ between -1 and 0. Therefore, $\hat{V}(\omega) \leqq -2$, and, in particular, $1+\hat{V}(\omega) < 0$, and $[\hat{V}^2(\omega)+2\hat{V}(\omega)]^{1/2}$ is real and non-negative. Hence,

$$|1+\hat{V}(\omega) - [\hat{V}^2(\omega)+2\hat{V}(\omega)]^{1/2}| \geqq |1+\hat{V}(\omega) + [\hat{V}^2(\omega)+2\hat{V}(\omega)]^{1/2}| ,$$

and $$\hat{z} = 1+\hat{V}(\omega) + [\hat{V}^2(\omega)+2\hat{V}(\omega)]^{1/2} , \quad (\rho_1\rho_2<0) . \tag{2.5.11}$$

To evaluate the integral in (2.5.5), we will use the saddle-point approximation,

$$f(\omega) \sim \left[\frac{-4\pi\chi(\omega,\hat{\xi})}{(n-3)\chi''(\omega,\hat{\xi})}\right]^{1/2} \theta(\omega,\hat{\xi})[\chi(\omega,\hat{\xi})]^{(n-3)/2} , \tag{2.5.12}$$

where $\chi''(\omega,\hat{\xi}) = \dfrac{\partial^2\chi}{\partial\xi^2}(\omega,\hat{\xi})$.

Considering the case, $\rho_1\rho_2 > 0$, we have already shown that V attains a unique minimum at $\xi = \hat{\xi}(\omega)$, and V tends to ∞ as ξ tends to ω^2 or to ∞. Furthermore,

$$\frac{d\hat{z}}{dV} = 1 - \frac{1+V}{[V^2+2V]^{1/2}} \neq 0, \tag{2.5.13}$$

$$\begin{aligned} \hat{z} &= 1+V - [V^2+2V]^{1/2} \\ &= 1+V - V[1+(2/V)]^{1/2} \qquad \text{for } V>0 \\ &= 1+V - V[1+(1/V)+O(V^{-2})] \\ &= O(V^{-1}) , \end{aligned}$$

and $\hat{z} \to 0$ as $V \to \infty$.

Hence, $\chi(\omega,\xi) = \hat{z}/\rho_1\rho_2 \to 0$ as $\xi \to \omega^2$ and as $\xi \to \infty$, and attains a maximum at $\xi = \xi(\omega)$. Therefore, the saddle-point method is applicable.

For the case, $\rho_1\rho_2 < 0$, we have already shown that V attains a unique maximum at $\xi = \hat{\xi}(\omega)$ and that V tends to $-\infty$ as ξ tends to ω^2 or to ∞. But now $V < -2 < 0$,

$$\frac{d\hat{z}}{dV} = 1 + \frac{1+V}{[V^2+2V]^{1/2}} \neq 0 \ , \tag{2.5.14}$$

$$\begin{aligned}\hat{z} &= 1+V + [V^2+2V]^{1/2} \\ &= 1+V + |V|[1+(2/V)]^{1/2} \\ &= 1+V - V[1+(1/V)+O(V^{-2})] \\ &= O(V^{-1}),\end{aligned}$$

and $z \to 0$ as $V \to \infty$.

Hence, again $\chi(\omega,\xi) \to 0$ as $\xi \to \omega^2$ and as $\xi \to \infty$, and attains a maximum at $\xi = \hat{\xi}(\omega)$.

Evaluating the derivative in (2.5.12),

$$\frac{\partial\chi}{\partial\xi} = \frac{\partial\chi}{\partial V}\frac{\partial V}{\partial\xi} \ ,$$

$$\frac{\partial^2\chi}{\partial\xi^2} = \frac{\partial^2\chi}{\partial V^2}\left(\frac{\partial V}{\partial\xi}\right)^2 + \frac{\partial\chi}{\partial V}\frac{\partial^2 V}{\partial\xi^2} \ ,$$

and $$\left.\frac{\partial^2\chi}{\partial\xi^2}\right|_{\xi=\hat{\xi}} = \frac{\partial\chi}{\partial V}\left.\frac{\partial^2 V}{\partial\xi^2}\right|_{\xi=\hat{\xi}} \quad . \tag{2.5.15}$$

From (2.5.7), using the fact that $\left.\frac{\partial V}{\partial\xi}\right|_{\xi=\hat{\xi}} = 0$,

$$\begin{aligned}\frac{\partial^2 V}{\partial\xi^2}(\hat{\xi}(\omega),\omega) &= [8\rho_1\rho_2(\hat{\xi}-\omega^2)^4]^{-1}\{(\hat{\xi}-\omega^2)^2[(1-\rho_2)^2[\hat{\xi}(1+\rho_2)^2+(1+\rho_1)^2] \\ &\quad + (1+\rho_2)^2[\hat{\xi}(1-\rho_2)^2+(1-\rho_1)^2] + 2(\hat{\xi}-\omega^2)(1-\rho_2^2)^2 \\ &\quad -2\hat{\xi}(1-\rho_2^2)^2 - (1-\rho_1)^2(1+\rho_2)^2 - (1+\rho_1)^2(1-\rho_2)^2]-0\} \\ &= [8\rho_1\rho_2(\hat{\xi}-\omega^2)^4]^{-1}\{(\hat{\xi}-\omega^2)^2[2(\hat{\xi}-\omega^2)(1-\rho_2^2)^2]\} \\ &= \frac{(1-\rho_2^2)^2}{4\rho_1\rho_2(\hat{\xi}-\omega^2)} \quad .\end{aligned} \tag{2.5.16}$$

Therefore, using (2.5.12), (2.5.13), and (2.5.14) ,

$$\chi''(\omega,\hat{\xi}) = \frac{1}{\rho_1\rho_2}\left[1-\operatorname{sgn}(\rho_1\rho_2)\frac{\hat{V}(\omega)+1}{[\hat{V}^2(\omega)+2\hat{V}(\omega)]^{1/2}}\right]\cdot\frac{(1-\rho_2^2)^2}{4\rho_1\rho_2(\hat{\xi}-\omega^2)} \quad , \tag{2.5.17}$$

and
$$\begin{aligned}\frac{-\chi(\omega,\hat{\xi})}{\chi''(\omega,\hat{\xi})} &= \frac{4|\rho_1\rho_2|(\hat{\xi}-\omega^2)[\hat{V}^2(\omega)+2\hat{V}(\omega)]^{1/2}}{(1-\rho_2^2)^2} \\ &= \frac{4|\rho_1\rho_2|[\hat{V}^2(\omega)+2\hat{V}(\omega)]^{1/2}\left[\left[\omega^2+\left(\frac{1-\rho_1}{1-\rho_2}\right)^2\right]\left[\omega^2+\left(\frac{1+\rho_1}{1+\rho_2}\right)^2\right]\right]^{1/2}}{(1-\rho_2^2)^2} \quad .\end{aligned} \tag{2.5.18}$$

Using (2.5.4), (2.5.12), and (2.5.18) ,

$$f(\omega) \sim \{16\pi|\rho_1\rho_2|[\hat{V}^2(\omega)+2\hat{V}(\omega)]^{1/2}[\omega^2+(1-\rho_1)^2/(1-\rho_2)^2]^{1/2}$$
$$\times\ [\omega^2+(1+\rho_1)^2/(1+\rho_2)^2]^{1/2}/[(n-3)(1-\rho_2^2)^2]\}^{1/2}$$
$$\times\ \frac{2(1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}n(n-2)}{\pi(n-4)} \quad \frac{(1-\hat{z}^2)(1-\hat{z})^2}{(1-\rho_1\rho_2\hat{z})} \quad \left(\frac{\hat{z}}{\rho_1\rho_2}\right)^{(n-3)/2}$$
$$\times\ [\hat{\xi}(1-\rho_2^2)+(1-\rho_1^2)]^{-1}[\hat{\xi}(1+\rho_2^2)+(1+\rho_1^2)]^{-1}$$
$$\times\ \{\hat{\xi}^2(1-\rho_2^2)^2+(1-\rho_1^2)^2+2\hat{\xi}[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\}^{-1/2}\ . \tag{2.5.19}$$

It appears, unfortunately, that this final expression cannot be simplified significantly. It was found to be difficult to renormalize analytically, without considerably more approximations. Therefore, a numerical approach was used. Thus, the final form of the approximation of the density function of b_1 is

$$f(b_1) = \frac{8n(n-2)}{\sqrt{\pi}(n-4)\sqrt{n-3}} \quad \frac{\sqrt{|\rho_1\rho_2|}(1-\rho_1^2)^{1/2}}{(1-\rho_2^2)^{1/2}} \quad [\hat{V}^2(b_1)+2\hat{V}(b_1)]^{1/4}$$
$$\times \left[\left[b_1^2 + \left(\frac{1-\rho_1}{1-\rho_2}\right)^2\right]\left[\ b_1^2 + \left(\frac{1+\rho_1}{1+\rho_2}\right)^2\ \right]\right]^{1/4}$$
$$\times\ \frac{(1-\hat{z}^2)(1-\hat{z})^2}{(1-\rho_1\rho_2\hat{z})} \cdot \left(\frac{\hat{z}}{\rho_1\rho_2}\right)^{(n-3)/2}$$
$$\times\ [\hat{\xi}(1-\rho_2^2)+(1-\rho_1^2)]^{-1}[\hat{\xi}(1+\rho_2^2)+(1+\rho_1^2)]^{-1}$$
$$\times\ \{\hat{\xi}^2(1-\rho_2^2)^2+(1-\rho_1^2)^2+2\hat{\xi}[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\ ^{-1/2}$$
$$+\ O(n^{-1})\ , \tag{2.5.20}$$

where $\hat{z} = 1+\hat{V}(b_1)-\mathrm{sgn}(\rho_1\rho_2)[\hat{V}^2(b_1)+2\hat{V}(b_1)]^{1/2}$, (2.5.21)

$$\hat{V}(b_1) = \frac{(1-\rho_2)^2[\hat{\xi}(1+\rho_2)^2+(1+\rho_1)^2] + (1+\rho_2)^2[\hat{\xi}(1-\rho_2)^2+(1-\rho_1)^2]}{8\rho_1\rho_2}, \tag{2.5.22}$$

and

$$\hat{\xi} = \hat{\xi}(b_1) = b_1^2 + \left[\left[b_1^2 + \left(\frac{1-\rho_1}{1-\rho_2}\right)^2\right]\left[b_1^2 + \left(\frac{1+\rho_1}{1+\rho_2}\right)^2\right]\right]^{1/2}. \tag{2.5.23}$$

As in the derivation for the approximate joint density, $h^*(b_1^*,b_2^*)$, of the two regression coefficients with fitted means, the approximate marginal density, $f^*(b_1^*)$, is obtained from $f(b_1)$ by substituting $(n-1)$ for n, and multiplying $f(b_1)$ by, from (2.4.38),

$$\left.\frac{(1+\hat{z})[b_1(1-\rho_2)^2+b_2(1-\rho_1)^2]}{(1-\rho_1)(1-\rho_2)[b_1(1+\rho_2^2)+b_2(1+\rho_1^2)]}\right|_{\xi=\hat{\xi}}$$

$$= \frac{(1+\hat{z})[\hat{\xi}(1-\rho_2)^2+(1-\rho_1)^2]}{(1-\rho_1)(1-\rho_2)[\hat{\xi}(1+\rho_2^2)+(1+\rho_1^2)]}\ . \tag{2.5.24}$$

Thus, for fitted means,

$$f^*(b_1^*) = \frac{8(n-1)(n-3)}{\sqrt{\pi}(n-5)\sqrt{n-4}}\ \frac{\sqrt{|\rho_1\rho_2|}(1+\rho_1)^{1/2}}{(1-\rho_1)^{1/2}(1-\rho_2)^{3/2}(1+\rho_2)^{1/2}}$$

$$\times\ [\hat{V}(b_1^*)+2\hat{V}(b_1^*)]^{1/4}$$

$$\times \left[\left[b_1^{*2} + \left(\frac{1-\rho_1}{1-\rho_2}\right)^2\right]\left[b_1^{*2} + \left(\frac{1+\rho_1}{1+\rho_2}\right)^2\right]\right]^{1/4}$$

(continued)

$$\times \frac{(1-\hat{z}^2)^2(1-\hat{z})}{(1-\rho_1\rho_2\hat{z})} \cdot \left(\frac{\hat{z}}{\rho_1\rho_2}\right)^{(n-4)/2}$$

$$\times\ [\hat{\xi}(1-\rho_2^2)+(1-\rho_1^2)]^{-1}[\hat{\xi}(1-\rho_2)^2+(1-\rho_1)^2][\hat{\xi}(1+\rho_2^2)+(1+\rho_1^2)]^{-2}$$

$$\times\ \{\hat{\xi}^2(1-\rho_2^2)^2+(1-\rho_1^2)^2+2\hat{\xi}[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\}^{-1/2}$$

$$+\ O(n^{-1})\ , \qquad (2.5.25)$$

where $\hat{z} = 1+\hat{V}(b_1^*)-\mathrm{sgn}(\rho_1\rho_2)[\hat{V}^2(b_1^*)+2\hat{V}(b_1^*)]^{1/2}$, (2.5.26)

$$\hat{V}(b_1^*) = \frac{(1-\rho_2)^2[\hat{\xi}(1+\rho_2)^2+(1+\rho_1)^2] + (1+\rho_2)^2[\hat{\xi}(1-\rho_2)^2+(1-\rho_1)^2]}{8\rho_1\rho_2}\ , \qquad (2.5.27)$$

and

$$\hat{\xi} = \hat{\xi}(b_1^*) = b_1^{*2} + \left[\left[b_1^{*2} + \left(\frac{1-\rho_1}{1-\rho_2}\right)^2\right]\left[b_1^{*2} + \left(\frac{1+\rho_1}{1+\rho_2}\right)^2\right]\right]^{1/2} . \qquad (2.5.28)$$

Note that both f and f^* are, to the degree of accuracy given, even functions, and hence, as conjectured in the introduction, b_1 and b_1^* are (approximately) symmetrically distributed about 0.

To show that equations (2.5.20), and (2.5.25) reduce to the corresponding classical densities, (1.2.12) and (1.2.16), we first note that, substituting $b_1 = \omega$ in (2.5.10), and letting ρ_1 and ρ_2 tend to 0,

$$\hat{V}(b_1) \to (8\rho_1\rho_2)^{-1}[2(b_1^2+1) + 2b_1^2 + 2] = \frac{b_1^2+1}{2\rho_1\rho_2} . \qquad (2.5.29)$$

Using equation (2.5.21), considering $\hat{z}$ as a function of $\hat{V}=\hat{V}(b_1)$, for $\hat{V}$ $(=\mathrm{sgn}(\rho_1\rho_2)|\hat{V}|)$ large,

$$\begin{aligned}\hat{z}^{-1} &= 1+\hat{V} +\mathrm{sgn}(\rho_1\rho_2)[\hat{V}^2+2\hat{V}]^{1/2} \\ &= 1+\hat{V} +\mathrm{sgn}(\rho_1\rho_2)|\hat{V}|[1+(2/\hat{V})]^{1/2} \\ &\approx 1+\hat{V} +\hat{V}[1+(1/\hat{V})] \\ &= 1+\hat{V}+\hat{V}+1 \\ &= 2\hat{V}+2 ,\end{aligned}$$

$$\text{and } \hat{z} \approx \frac{1}{2\hat{V}+2} . \qquad (2.5.30)$$

But, from equation (2.5.29), $\hat{V}$ is large for small $\rho_1\rho_2$,

$$\text{and } \hat{z} \approx \frac{\rho_1\rho_2}{b_1^2+1+2\rho_1\rho_2} \approx \frac{\rho_1\rho_2}{b_1^2+1} \quad \text{when } \rho_1\rho_2 \text{ is small} . \qquad (2.5.31)$$

Examining the remaining factors in (2.5.21), for $\rho_1\rho_2$ small,

$$|\rho_1\rho_2|[\hat{V}^2+2\hat{V}]^{1/2} \approx |\rho_1\rho_2||\hat{V}| \approx \frac{b_1^2+1}{2} ,$$

$$\left[\left[b_1^2 + \left(\frac{1-\rho_1}{1-\rho_2}\right)^2\right]\left[b_1^2 + \left(\frac{1+\rho_1}{1+\rho_2}\right)^2\right]\right]^{1/2} \approx b_1^2+1$$

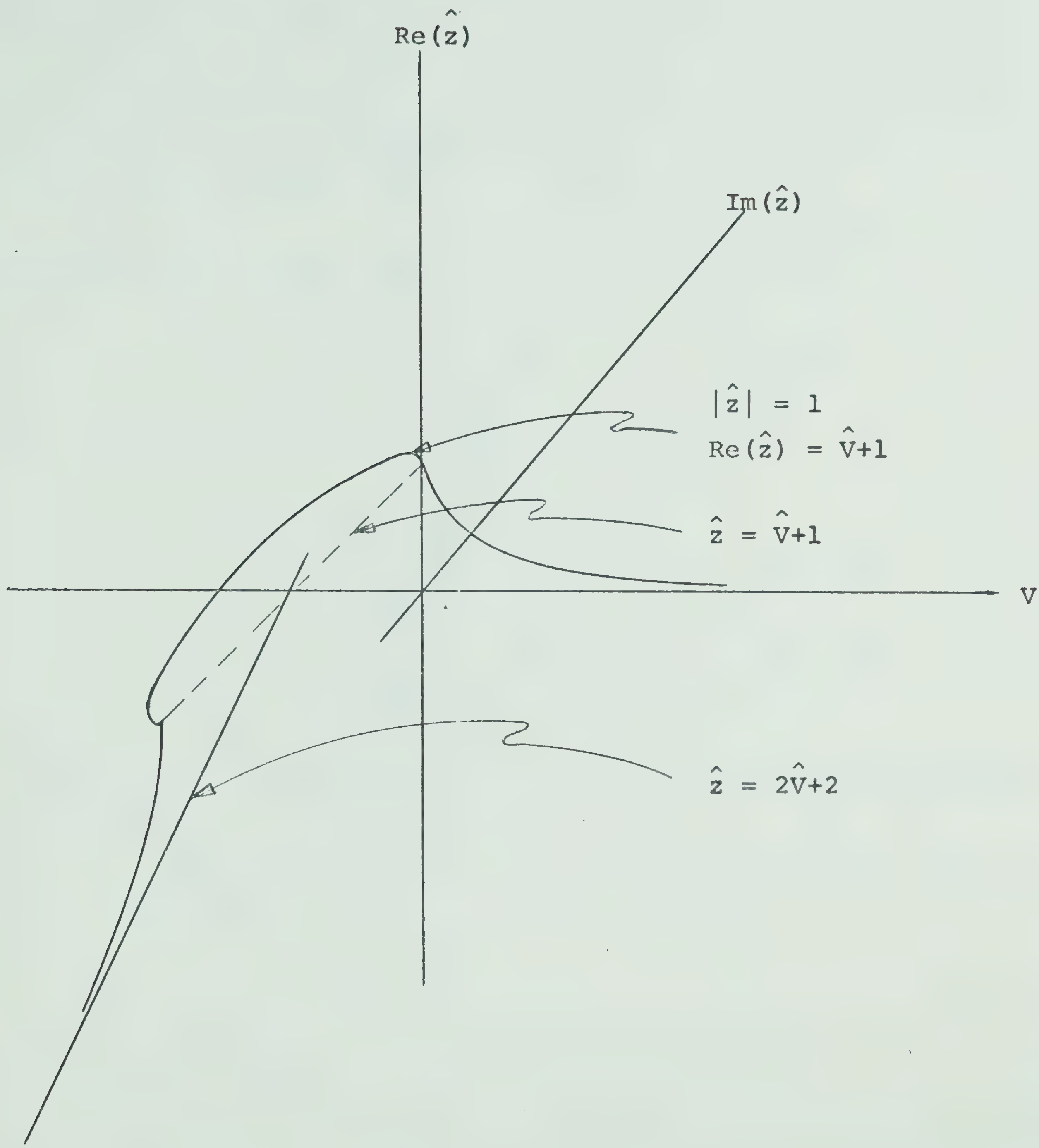


Figure 3: The Graph of $\hat{z} = 1+\hat{V} - (\hat{V}^2+2\hat{V})^{1/2}$.

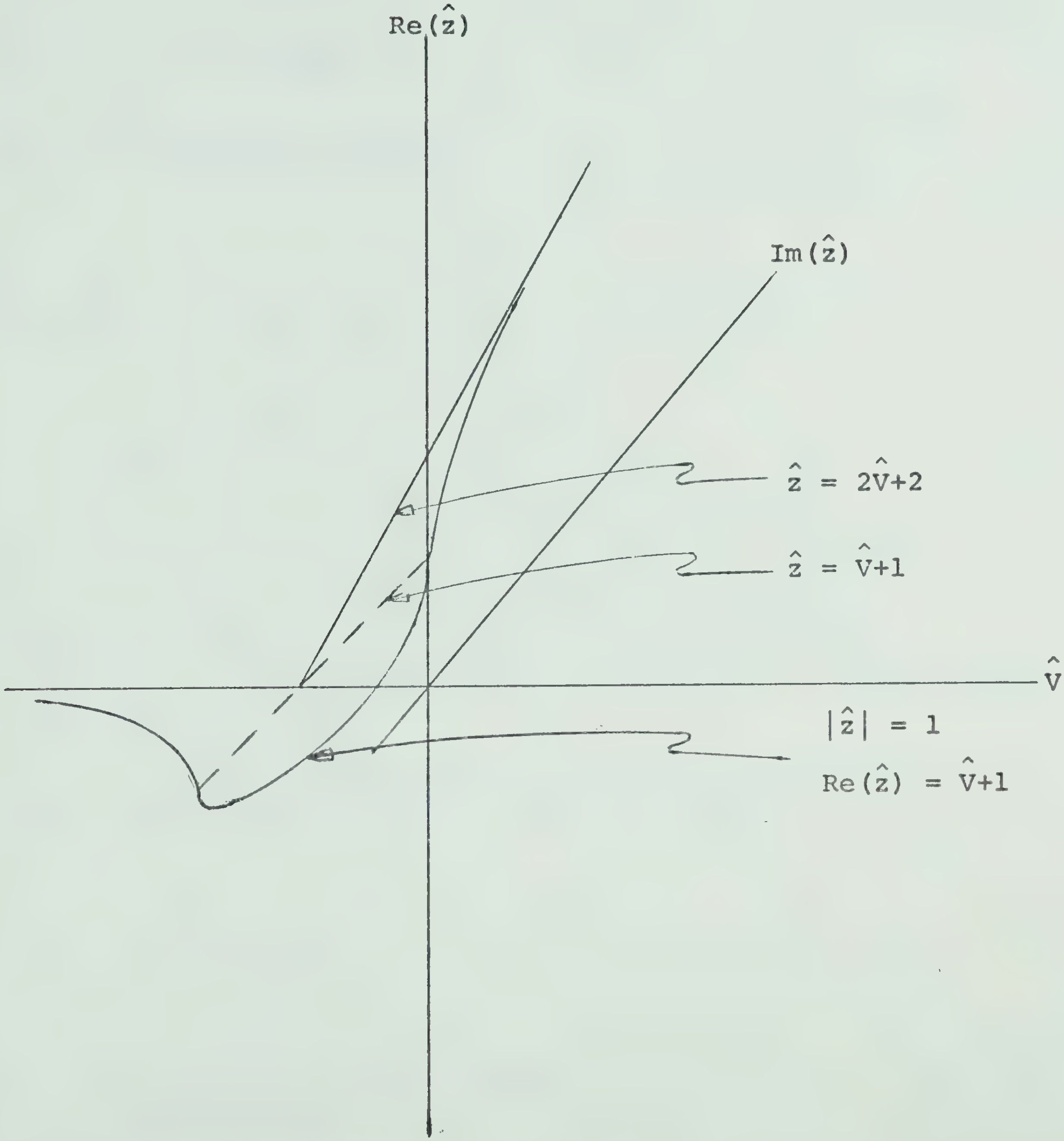


Figure 4: The Graph of $\hat{z} = 1+\hat{V} + (\hat{V}^2+2\hat{V})^{1/2}$.

$$(1-\hat{z}^2) \approx 1 ,$$

$$(1-\hat{z})^2 \approx 1 ,$$

$$(1-\rho_1\rho_2\hat{z}) \approx 1 ,$$

and, from equation (2.5.23),

$$\hat{\xi} \approx 2b_1^2+1 ,$$

and hence, $\hat{\xi}(1-\rho_2^2)+(1-\rho_1^2) \approx 2b_1^2+2 = 2(b_1^2+1)$,

$$\hat{\xi}(1+\rho_2^2)+(1+\rho_1^2) \approx 2(b_1^2+1) ,$$

and $\hat{\xi}^2(1-\rho_2^2)^2+(1-\rho_1^2)^2+2\hat{\xi}[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]$

$$\approx (2b_1^2+1)^2+1+2(2b_1^2+1) = 4(b_1^2+1)^2 .$$

Substituting these expressions into (2.5.20), we obtain that, as ρ_1 and ρ_2 tend to 0,

$$f(b_1) \to \frac{8n(n-2)}{\sqrt{\pi}(n-4)\sqrt{n-3}} \left(\frac{b_1^2+1}{2}\right)^{1/2} (b_1^2+1)^{1/2} \left(\frac{1}{b_1^2+1}\right)^{(n-3)/2}$$

$$\times \left(\frac{1}{2(b_1^2+1)}\right)^3$$

$$= \frac{n(n-2)}{\sqrt{2\pi}(n-4)\sqrt{n-3}} (1+b_1^2)^{-(n+1)/2} . \qquad (2.5.32)$$

Except for the constant term, this is identical to equation (1.2.12) for the classical density of b_1.

To evaluate the limit of $f^*(b_1^*)$ as ρ_1 and ρ_2 tend to 0, we note that $\hat{\xi}(1-\rho_2)^2+(1-\rho_1)^2 \approx 2(b_1^{*2}+1)$,and that the limiting relations derived for the previous case are still valid, replacing b_1 by b_1^*. Hence, from equation (2.5.25),

$$f(b_1^*) \to \frac{8(n-1)(n-3)}{\sqrt{\pi}(n-5)\sqrt{n-4}} \left(\frac{b_1^{*2}+1}{2}\right)^{1/2} (b_1^{*2}+1)^{1/2} \left(\frac{1}{b_1^{*2}+1}\right)^{(n-4)/2}$$

$$\times\left(\frac{1}{2(b_1^{*2}+1)}\right)^3$$

$$= \frac{(n-1)(n-3)}{\sqrt{2\pi}(n-5)\sqrt{n-4}} (1+b_1^{*2})^{-n/2} \quad . \qquad (2.5.33)$$

Again, except for the constant term, this is identical to equation (1.2.16) for the classical density of b_1^*.

2.6 The Approximate Variances

Since f, the density of b_1, is even, $E(b_1) = 0$,

and $\mathrm{Var}(b_1) = \int b_1^2 f(b_1)\,db_1$.

Using equation (2.5.20), this becomes

$$\mathrm{Var}(b_1) \sim \int \frac{8n(n-2)}{\sqrt{\pi}(n-4)\sqrt{n-3}} \frac{\sqrt{|\rho_1\rho_2|}(1-\rho_1^2)^{1/2}}{(1-\rho_2^2)^{1/2}} [\hat{V}^2+2\hat{V}]^{1/4}$$

$$\times \left[\left[b_1^2 + \left(\frac{1-\rho_1}{1-\rho_2}\right)^2\right]\left[b_1^2 + \left(\frac{1+\rho_1}{1+\rho_2}\right)^2\right]\right]^{1/4}$$

$$\times \frac{(1-\hat{z}^2)(1-\hat{z})^2 b_1^2}{(1-\rho_1\rho_2\hat{z})} \left(\frac{\hat{z}}{\rho_1\rho_2}\right)^{(n-3)/2}$$

$$\times [\hat{\xi}(1-\rho_2^2)+(1-\rho_1^2)]^{-1}[\hat{\xi}(1+\rho_2^2)+(1+\rho_1^2)]^{-1}$$

$$\times \{\hat{\xi}^2(1-\rho_2^2)^2+(1-\rho_1^2)^2+2\hat{\xi}[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\}^{-1/2}$$

$$\times db_1 . \tag{2.6.1}$$

The factor, $(\hat{z}/\rho_1\rho_2)^{(n-3)/2}$, may be rewritten, using (2.5.20),

as $\exp\{[(n-3)/2]\ln[(1+\hat{V}-\mathrm{sgn}(\rho_1\rho_2)(\hat{V}^2+2\hat{V})^{1/2})/(\rho_1\rho_2)]\}$

$$= \exp\{[(n-3)/2]\phi(b_1^2)\} , \tag{2.6.2}$$

where $\phi(b_1^2) = \ln[(1+\hat{V}-\mathrm{sgn}(\rho_1\rho_2)(\hat{V}^2+2\hat{V})^{1/2})/(\rho_1\rho_2)]$. (2.6.3)

(By (2.5.21) and (2.5.22), $\hat{V}$ is an even function, and hence, ϕ is well defined.)

For b_1 close to 0,

$$\phi(b_1^2) \sim \phi(0) + b_1^2\phi'(0) \;. \tag{2.6.4}$$

By (2.5.21), $\hat{\xi}(0) = \dfrac{1-\rho_1^2}{1-\rho_2^2}$,

$$\hat{V}(0) = \frac{(1-\rho_1^2)(1-\rho_2^2)+(1-\rho_2)^2(1+\rho_1)^2+(1-\rho_1^2)(1-\rho_2^2)+(1-\rho_1)^2(1-\rho_2)^2}{8\rho_1\rho_2}$$

$$= \frac{[(1-\rho_2)(1+\rho_1)+(1-\rho_1)(1+\rho_2)]^2}{8\rho_1\rho_2}$$

$$= \frac{(1-\rho_1\rho_2)^2}{2\rho_1\rho_2} \;, \tag{2.6.5}$$

and

$$[\hat{V}^2(0)+2\hat{V}(0)]^{1/2} = \frac{[(1-\rho_1\rho_2)^4+4\rho_1\rho_2(1-\rho_1\rho_2)^2]^{1/2}}{2|\rho_1\rho_2|}$$

$$= \frac{(1-\rho_1\rho_2)[(1-\rho_1\rho_2)^2+4\rho_1\rho_2]^{1/2}}{2|\rho_1\rho_2|}$$

$$= \frac{(1-\rho_1\rho_2)(1+\rho_1\rho_2)}{2|\rho_1\rho_2|} = \frac{(1-\rho_1^2\rho_2^2)}{2|\rho_1\rho_2|} \;. \tag{2.6.6}$$

Thus $\left. 1+\hat{V}-\mathrm{sgn}(\rho_1\rho_2)[\hat{V}^2+2\hat{V}]^{1/2}\right|_{b_1=0}$

$$= \frac{2\rho_1\rho_2+(1-\rho_1\rho_2)^2-(1-\rho_1^2\rho_2^2)}{2\rho_1\rho_2}$$

$$= \frac{2\rho_1\rho_2+1-2\rho_1\rho_2+\rho_1^2\rho_2^2-1+\rho_1^2\rho_2^2}{2\rho_1\rho_2} = \rho_1\rho_2 \;, \tag{2.6.7}$$

and $\phi(0) = \ln[(\rho_1\rho_2)/(\rho_1\rho_2)] = \ln(1) = 0.$ (2.6.8)

Furthermore,

$$\phi'(0) = \frac{\rho_1\rho_2}{1+\hat{V}(0)-\operatorname{sgn}(\rho_1\rho_2)[\hat{V}^2(0)+2\hat{V}(0)]^{1/2}}$$

$$\times \frac{\partial \exp\phi}{\partial V}(0) \quad \frac{\partial V}{\partial b_1^2}(0) \quad , \tag{2.6.9}$$

and, by (2.5.5),

$$\frac{\partial\hat{V}}{\partial b_1^2} = [8\rho_1\rho_2]^{-1}\left[(1-\rho_2^2)^2\left[\left[b_1^2 + \left(\frac{1-\rho_1}{1-\rho_2}\right)^2\right]\left[b_1^2 + \left(\frac{1+\rho_1}{1+\rho_2}\right)^2\right]\right]^{-1/2}\right.$$

$$\left. \times\left[\left(\frac{1-\rho_1}{1-\rho_2}\right)^2 + \left(\frac{1+\rho_1}{1+\rho_2}\right)^2\right] + 2(1-\rho_2^2)^2\right] \quad ,$$

and

$$\frac{\partial\hat{V}}{\partial b_1^2}(0) = [8\rho_1\rho_2]^{-1}\ \{[(1-\rho_2^2)/(1-\rho_1^2)][(1-\rho_1)^2(1+\rho_2)^2$$

$$+ (1+\rho_1)^2(1-\rho_2)^2] + 2(1-\rho_2^2)^2\}$$

$$= [8\rho_1\rho_2]^{-1}\ [(1-\rho_2^2)/(1-\rho_1^2)][(1-\rho_1)(1+\rho_2)+(1+\rho_1)(1-\rho_2)]^2$$

$$= [2\rho_1\rho_2]^{-1}[(1-\rho_2^2)/(1-\rho_1^2)][1-\rho_1\rho_2]^2 \quad . \tag{2.6.10}$$

Therefore, using (2.6.3), (2.6.6), (2.6.7), and (2.6.10) in (2.6.9),

$$\phi'(0) = 1 \times \{1-\operatorname{sgn}(\rho_1\rho_2)[\hat{V}(0)+1][\hat{V}^2(0)+2\hat{V}(0)]^{-1/2}$$

$$\times [2\rho_1\rho_2]^{-1}[(1-\rho_2^2)/(1-\rho_1^2)][1-\rho_1\rho_2]^2$$

$$= [-\operatorname{sgn}(\rho_1\rho_2)][(1-\rho_1^2\rho_2^2)/2|\rho_1\rho_2|]^{-1}$$

$$\times [2\rho_1\rho_2]^{-1}[(1-\rho_2^2)/(1-\rho_1^2)][1-\rho_1\rho_2]^2$$

(continued)

$$= - \frac{1-\rho_2^2}{1-\rho_1^2} \quad \frac{1-\rho_1\rho_2}{1+\rho_1\rho_2} \quad . \qquad (2.6.11)$$

Using (2.6.8) and (2.6.11) in (2.6.4),

$$\phi(b_1^2) \sim - \frac{1-\rho_2^2}{1-\rho_1^2} \cdot \frac{1-\rho_1\rho_2}{1+\rho_1\rho_2} \cdot b_1^2 \quad ,$$

and, by (2.6.2) ,

$$\left(\frac{\hat{z}}{\rho_1\rho_2}\right)^{(n-3)/2} \sim \exp -\left[\left(\frac{1-\rho_2^2}{1-\rho_1^2}\right)\left(\frac{1-\rho_1\rho_2}{1+\rho_1\rho_2}\right)\left(\frac{n-3}{2}\right) b_1^2\right]. \qquad (2.6.12)$$

Finally, (2.6.1) becomes

$$\mathrm{Var}(b_1) \sim \int \psi(b_1)\exp\left[- \left(\frac{1-\rho_2^2}{1-\rho_1^2}\right)\left(\frac{1-\rho_1\rho_2}{1+\rho_1\rho_2}\right)\left(\frac{n-3}{2}\right) b_1^2\right] db_1 \qquad (2.6.13)$$

where

$$\psi(b_1) = \left\{\frac{|\rho_1\rho_2|\,[\hat{v}^2+2\hat{v}]^{1/2}\left[\left[b_1^2 + \left(\frac{1-\rho_1}{1-\rho_2}\right)^2\right]\left[b_1^2 + \left(\frac{1+\rho_1}{1+\rho_2}\right)^2\right]\right]^{\frac{1}{2}}}{\pi(n-3)}\right\}^{\frac{1}{2}}$$

$$\times \frac{8(1-\rho_1^2)^{1/2} n(n-2)}{(1-\rho_2^2)^{1/2}(n-4)} \times \frac{(1-\hat{z}^2)(1-\hat{z})^2 b_1^2}{(1-\rho_1\rho_2\hat{z})}$$

$$\times [\hat{\xi}(1-\rho_2^2)+(1-\rho_1^2)]^{-1}[\hat{\xi}(1+\rho_2^2)+(1+\rho_1^2)]^{-1}$$

$$\times \{\hat{\xi}^2(1-\rho_2^2)^2+(1-\rho_1^2)^2+2\hat{\xi}[(\rho_1-\rho_2)^2+1-\rho_1\rho_2)^2]\}^{-1/2} .$$

Using the saddle-point approximation (see Daniels(1954) equation (3.1)),

$$\mathrm{Var}(b_1) \sim \left[\frac{2\pi(1-\rho_1^2)(1+\rho_1\rho_2)}{(n-3)(1-\rho_2^2)(1-\rho_1\rho_2)}\right]^{1/2}$$

$$\times \left[\psi(0) + \frac{(1-\rho_1^2)(1+\rho_1\rho_2)}{2(n-3)(1-\rho_2^2)(1-\rho_1\rho_2)}\,\psi''(0)\right] \quad . \qquad (2.6.14)$$

Clearly, $\psi(0) = 0$, and the only non-zero term in $\psi''(0)$ is that corresponding to differentiating the factor b_1^2 twice.

Furthermore, $\hat{\xi}(0) = \dfrac{1-\rho_1^2}{1-\rho_2^2}$. (see equation (2.5.23))

$$\text{Hence,}\quad \hat{\xi}(0)(1-\rho_2^2)+(1-\rho_1^2) = 2(1-\rho_1^2) \quad ,$$

$$\hat{\xi}(0)(1+\rho_2^2)+(1+\rho_1^2) = \frac{2(1-\rho_1^2\rho_2^2)}{(1-\rho_2^2)} \; ,$$

$$\text{and}\quad \hat{\xi}^2(1-\rho_2^2)^2+(1-\rho_1^2)^2+2\hat{\xi}[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]$$

$$= \frac{2(1-\rho_1^2)^2(1-\rho_2^2)+2(1-\rho_1^2)[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]}{(1-\rho_2^2)}$$

$$= \frac{4(1-\rho_1^2)(1-\rho_1\rho_2)^2}{(1-\rho_2^2)} \quad ,$$

and (2.6.14) becomes

$$\mathrm{Var}(b_1) = \frac{\sqrt{2\pi}(1-\rho_1^2)^{1/2}(1+\rho_1\rho_2)^{1/2}(1-\rho_1^2)(1+\rho_1\rho_2)(2)(4\sqrt{\pi})\sqrt{\rho_1\rho_2}}{\sqrt{n-3}\;(1-\rho_2^2)^{1/2}(1-\rho_1\rho_2)^{1/2}(2)\sqrt{n-3}(1-\rho_2^2)(1-\rho_1\rho_2)\sqrt{2\rho_1\rho_2}}$$

$$\times \frac{(1-\rho_1^2\rho_2^2)^{1/2}(1-\rho_1^2)^{1/2}(2)(1-\rho_1^2)^{1/2}(1-\rho_2^2)^{1/2}(1-\rho_1\rho_2)^2}{(n-3)(1-\rho_2^2)^{3/2}\pi(1-\rho_1^2\rho_2^2)(2)(1-\rho_1^2)(2)(1-\rho_1^2\rho_2^2)(2)}$$

$$\times \frac{(1-\rho_1^2\rho_2^2)(1-\rho_2^2)(1-\rho_2^2)^{1/2}n(n-2)}{(1-\rho_1^2)^{1/2}(1-\rho_1\rho_2)}$$

$$= \frac{1-\rho_1^2}{1-\rho_2^2}\cdot\frac{1+\rho_1\rho_2}{1-\rho_1\rho_2}\cdot\frac{n(n-2)}{(n-3)^2(n-4)}$$

$$\sim \frac{1-\rho_1^2}{1-\rho_2^2}\cdot\frac{1+\rho_1\rho_2}{1-\rho_1\rho_2}\cdot\frac{1}{n} \quad . \tag{2.6.15}$$

To find the variance of b_1^*, note that $f^*(b_1^*)$ was obtained from $f(b_1)$ by multiplying by expression (2.5.24) and changing n to $(n-1)$. The latter change has no effect on the asymptotic variance; the former multiplies $\psi(0)$ by expression (2.5.24), and hence, multiplies $\mathrm{Var}(b_1)$ by the same quantity.

Hence,

$$\mathrm{Var}(b_1^*) \sim \mathrm{Var}(b_1) \times \left.\frac{(1+\hat{z})[\hat{\xi}(1-\rho_2)^2+(1-\rho_1)^2]}{(1-\rho_1)(1-\rho_2)[\hat{\xi}(1+\rho_2^2)+(1+\rho_1^2)]}\right|_{b_1^*=0}$$

But, by (2.6.7), $\hat{z}\big|_{b_1^*=0} = \rho_1\rho_2$,

$$\text{and} \quad \hat{\xi}(0)(1+\rho_2^2)+(1+\rho_1^2) = \frac{2(1-\rho_1^2\rho_2^2)}{(1-\rho_2^2)} ,$$

$$\begin{aligned}\text{and} \quad \hat{\xi}(0)(1-\rho_2)^2+(1-\rho_1)^2 &= \frac{(1-\rho_1^2)}{(1-\rho_2^2)}(1-\rho_2)^2+(1-\rho_1)^2 \\ &= \frac{(1-\rho_1)[(1+\rho_1)(1-\rho_2)+(1-\rho_1)(1+\rho_2)]}{1+\rho_2} \\ &= \frac{2(1-\rho_1)(1-\rho_1\rho_2)}{1+\rho_2}\end{aligned}$$

Finally,

$$\begin{aligned}\mathrm{Var}(b_1^*) &\sim \frac{(1-\rho_1^2)}{(1-\rho_2^2)} \cdot \frac{(1+\rho_1\rho_2)}{(1-\rho_1\rho_2)} \cdot \frac{1}{n} \\ &\quad \times \frac{(1+\rho_1\rho_2)(2)(1-\rho_1)(1-\rho_1\rho_2)(1-\rho_2^2)}{(1-\rho_1)(1-\rho_2)(2)(1-\rho_1^2\rho_2^2)(1+\rho_2)} \\ &= \frac{1-\rho_1^2}{1-\rho_2^2} \cdot \frac{1+\rho_1\rho_2}{1-\rho_1\rho_2} \cdot \frac{1}{n} .\end{aligned} \tag{2.6.16}$$

the same expression as (2.6.15), for $\mathrm{Var}(b_1)$.

CHAPTER III

COMPUTATIONS AND CONCLUSIONS

3.1 The Calculation and Renormalization of Some Approximate Densities with Known Means

Using equations (2.5.20) through (2.5.23), noting the symmetry, one hundred ordinates of each of several densities were calculated. Since $f(b_1)$ was found to be negligible for all values of $|b_1|$ larger than 3, these ordinates were calculated for b_1 varying from -3 to +3. The integral, $\int f(b_1)db_1$, was then calculated numerically with a Simpson's rule routine. This quantity was printed out, and then used to renormalize $f(b_1)$. Renormalized graphs of $f(b_1)$ were then drawn using the "Autoplotter" package of the Computing Center.

A few values of the renormalizing constant have been displayed in Table 1. They exemplify the general trend for the constant to approach 1 with increasing n. The value of the constant for $\rho_1 = \rho_2 = 0$ and $n = 30$ indicates the accuracy of the numerical integration.

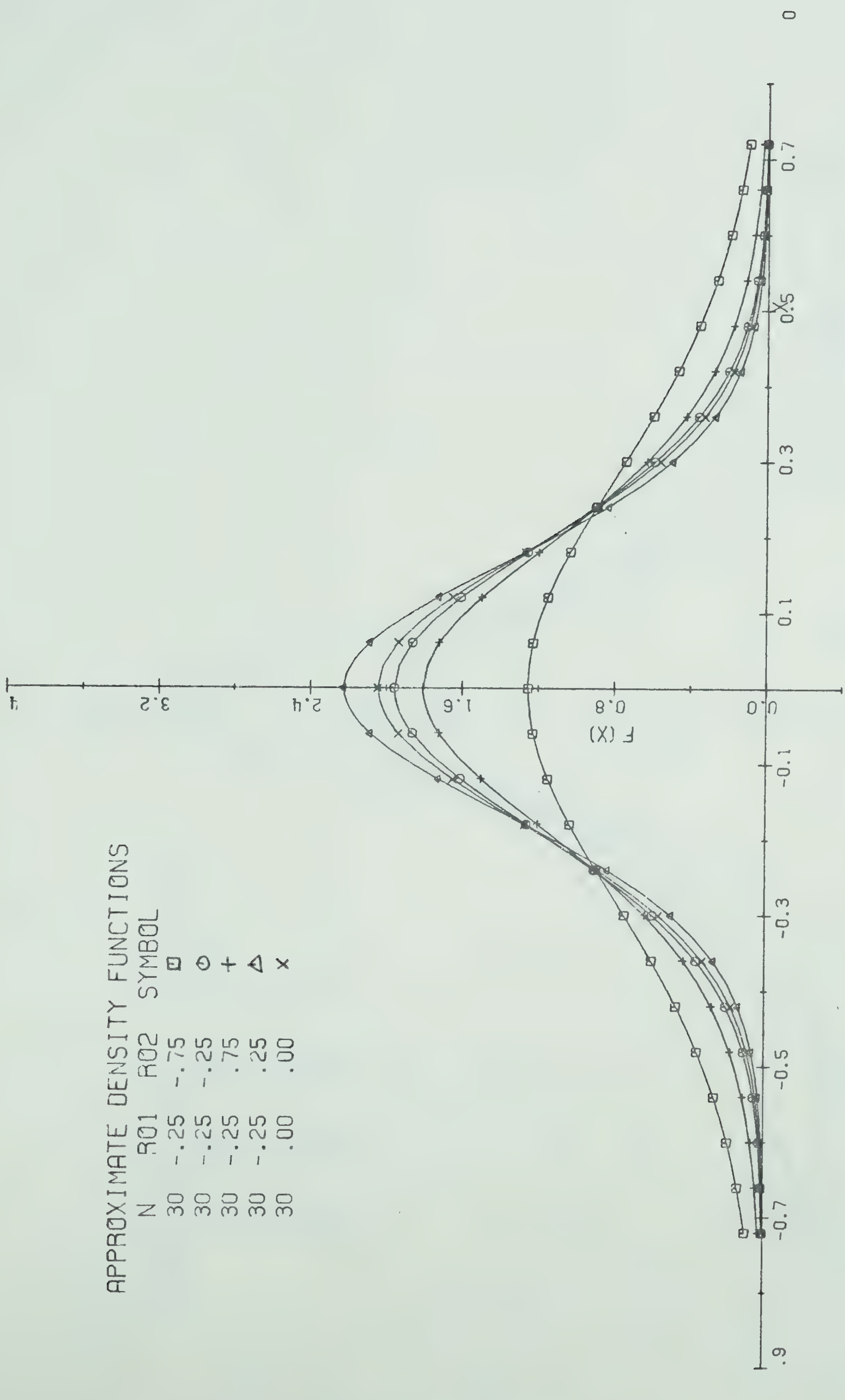
Figures 1(a) to (d) illustrate the behaviour of the density function for a sample size of 30, and for a representative selection of values of the auto-correlations, ρ_1 and ρ_2. This behaviour conforms to what would be expected by considering the types of scatter diagrams that are likely to occur. It

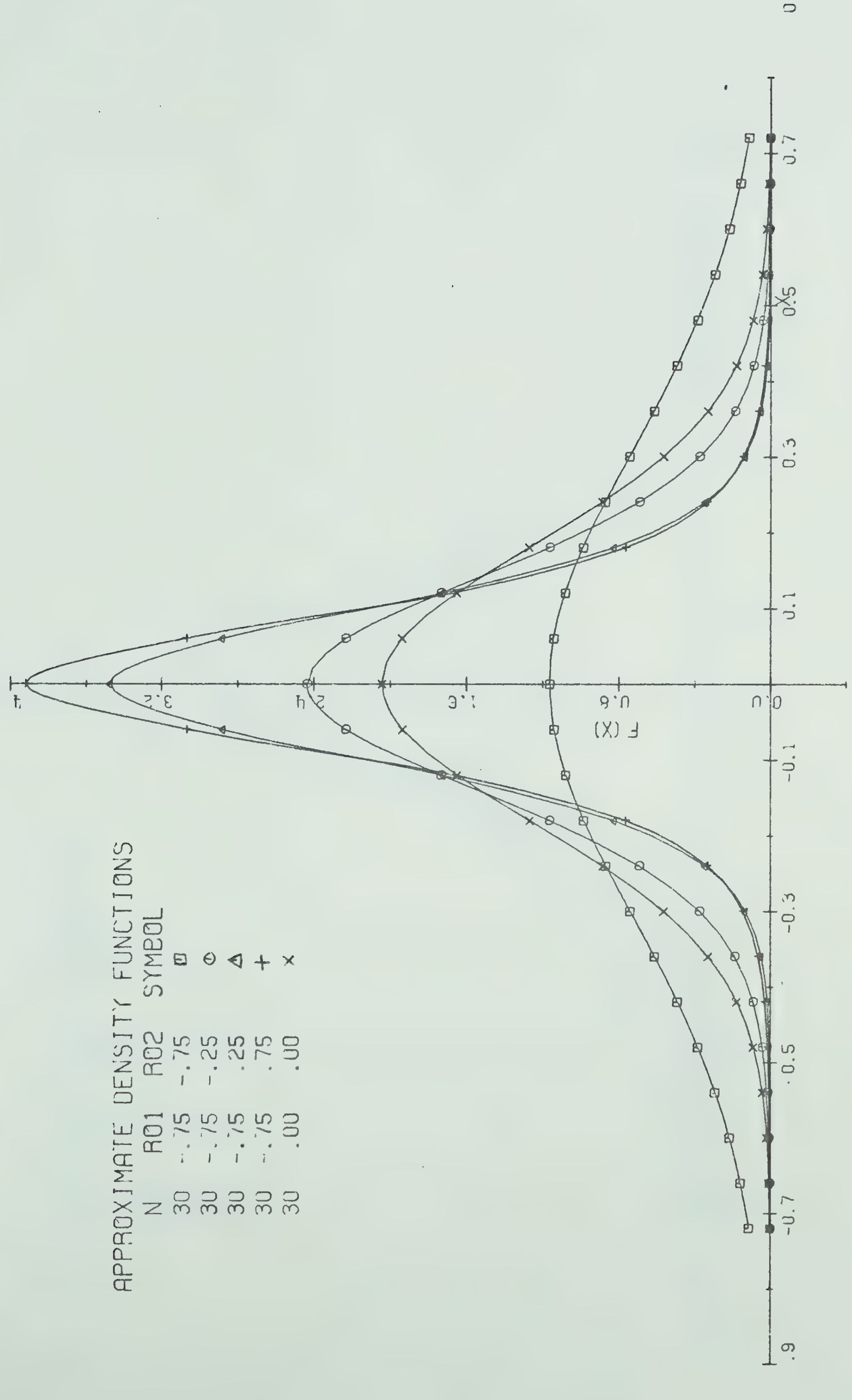
should also be noted that the density is unchanged if the signs of ρ_1 and ρ_2 are changed simultaneously, that the variance is largest for $\rho_1\rho_2$ large and positive, and that the variance is smallest for $\rho_1\rho_2$ large and negative. All three of these tendencies agree with the expression, (2.6.15), for the variance of b_1.

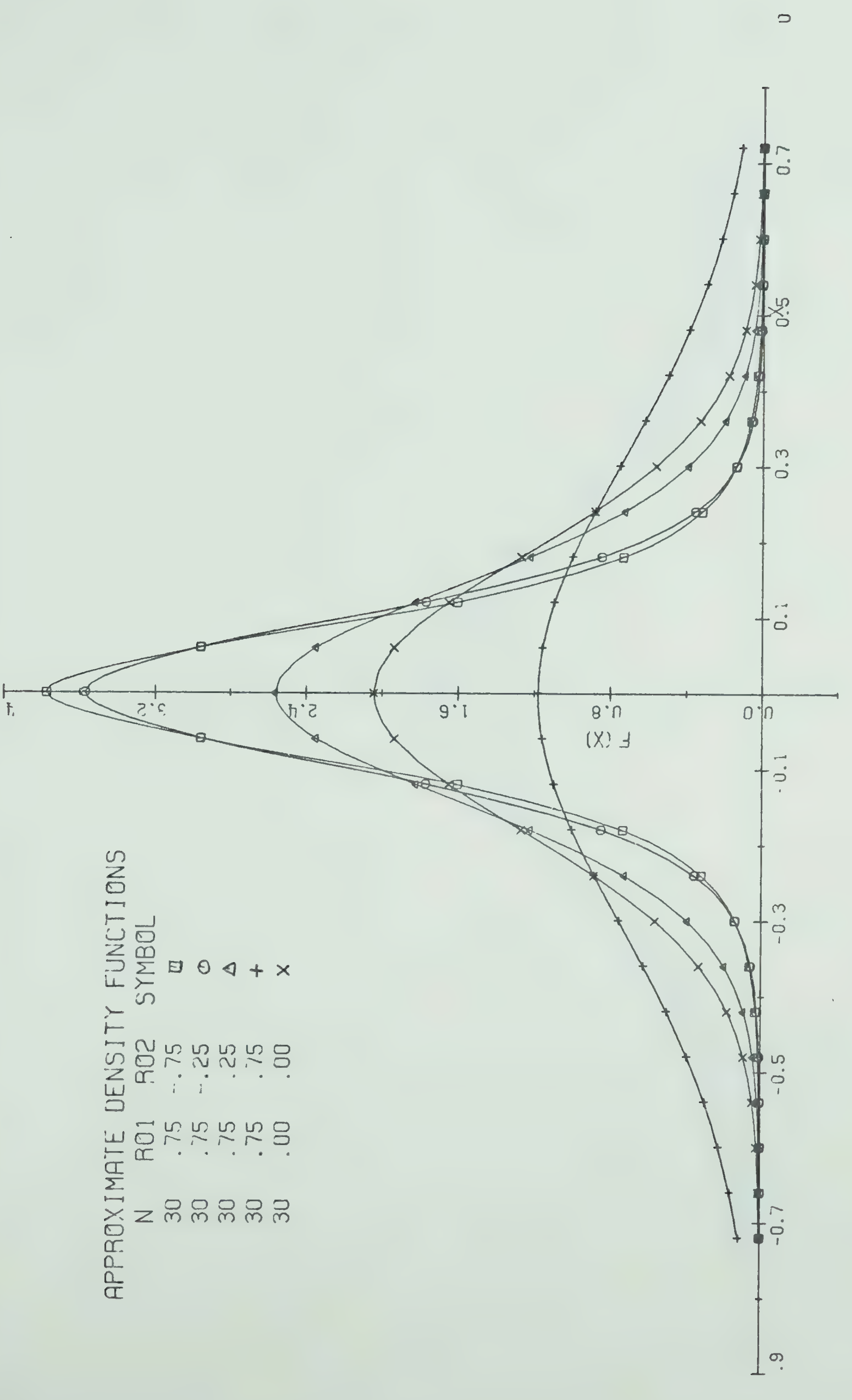
Figure 1(e) illustrates the predicted behaviour of the density for increasing values of n.

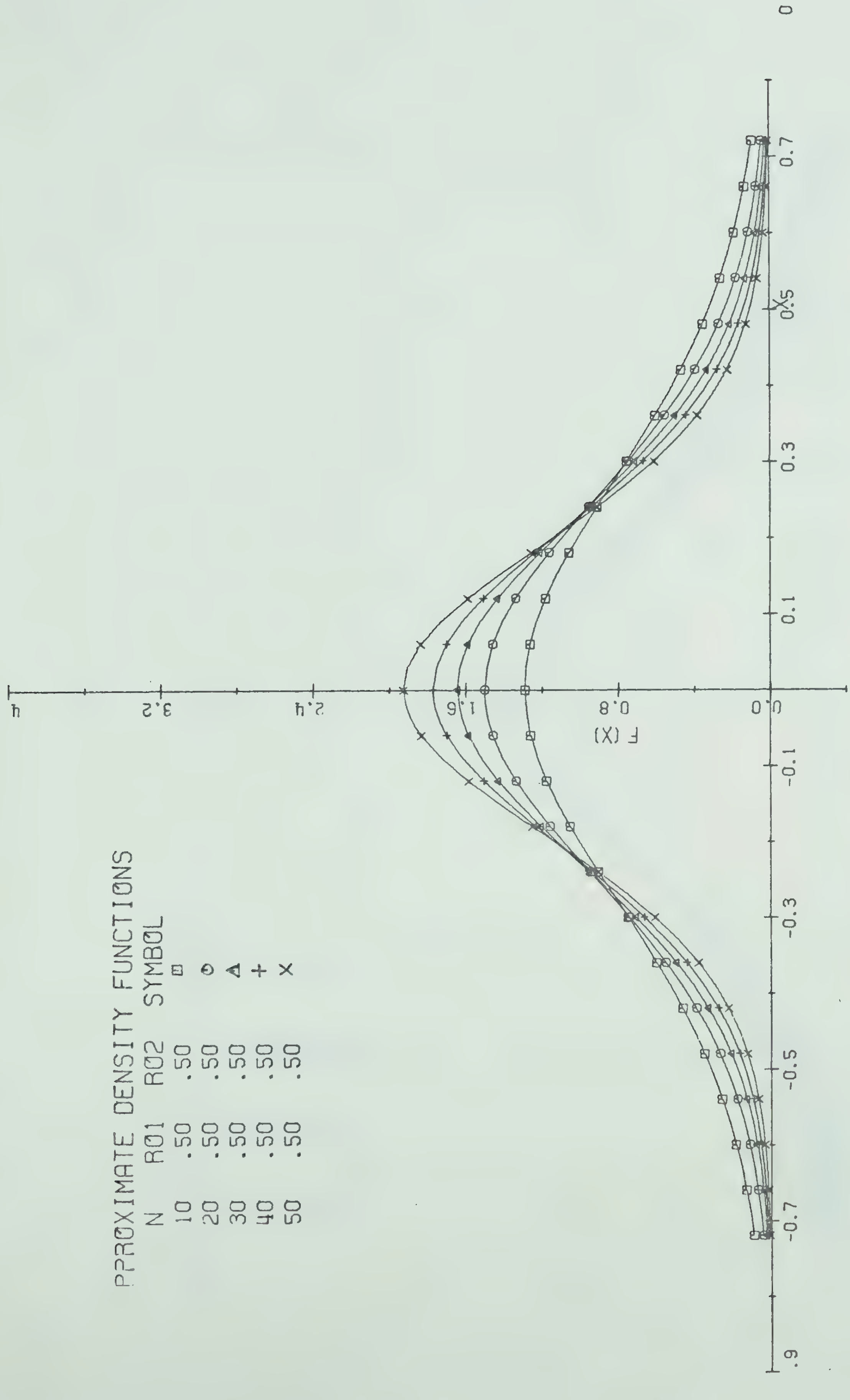
n	ρ_1	ρ_2	RC
10	0.50	0.50	1.56
20	0.50	0.50	1.23
30	0.50	0.50	1.16
40	0.50	0.50	1.13
50	0.50	0.50	1.11
30	-0.75	-0.75	1.13
30	-0.75	0.75	1.20
30	0.75	-0.75	1.24
30	0.75	0.75	1.11
30	0.0	0.0	1.04

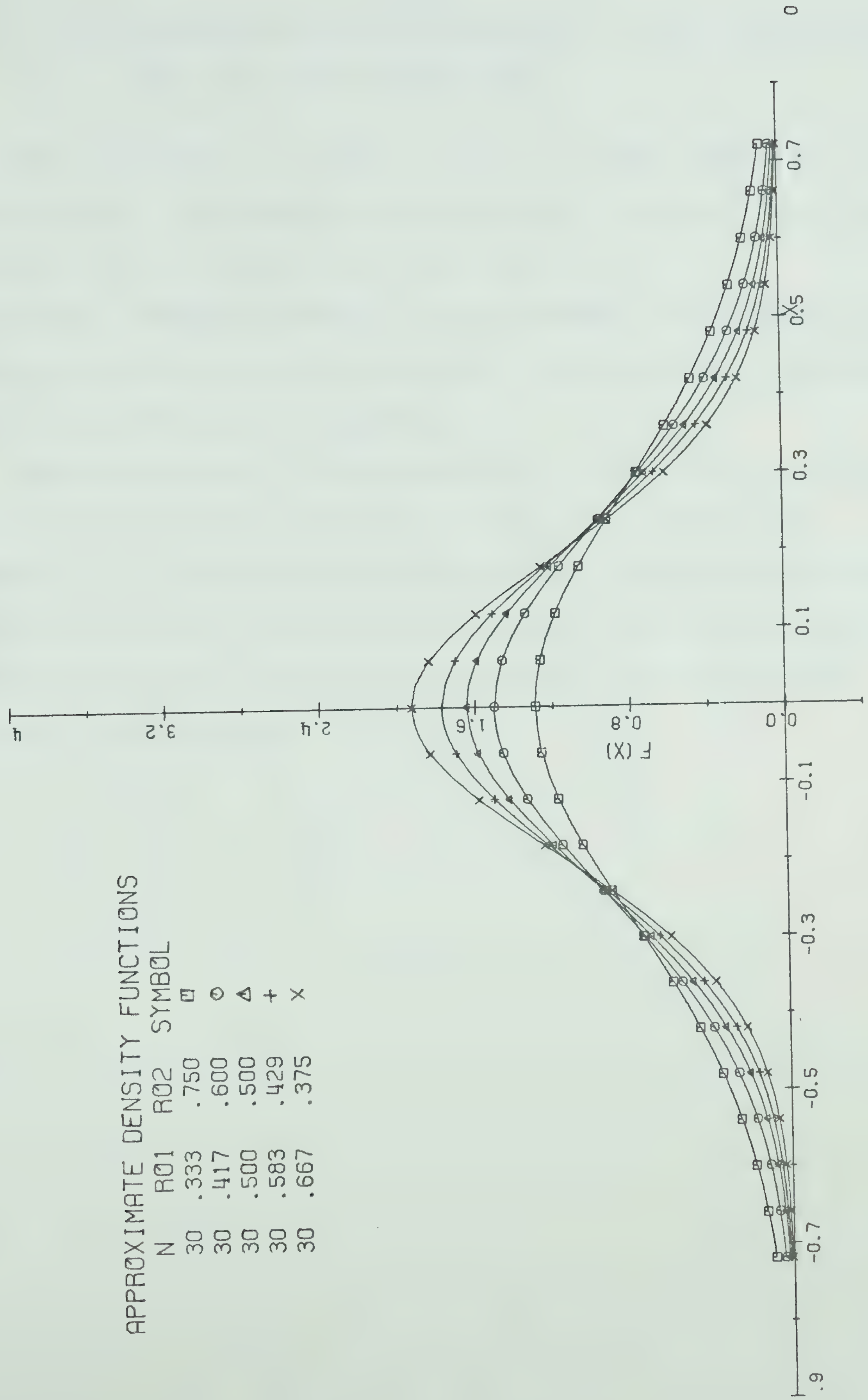
Table 1 : Renormalization Constants (RC) : Known Means Case


APPROXIMATE DENSITY FUNCTIONS
N RO1 RO2 SYMBOL
30 -.25 -.75 ⊡
30 -.25 -.25 ⊖
30 -.25 .75 +
30 -.25 .25 Δ
30 .00 .00 ×
F (X)
0.0
0.8
1.6
2.4
3.2
4
X
-0.9
-0.7
-0.5
-0.3
-0.1
0.1
0.3
0.5
0.7
0.9


APPROXIMATE DENSITY FUNCTIONS
N RO1 RO2 SYMBOL
30 -.75 -.75
30 -.75 -.25
30 -.75 .25
30 -.75 .75
30 .00 .00
F(X)
0.0
0.8
1.6
2.4
3.2
4
X
-0.7
-0.5
-0.3
-0.1
0.1
0.3
0.5
0.7


APPROXIMATE DENSITY FUNCTIONS
N R01 R02 SYMBOL
30 .75 -.75 □
30 .75 -.25 ○
30 .75 .25 △
30 .75 .75 +
30 .00 .00 ×
F(X)
X
0.0
0.8
1.6
2.4
3.2
4
-0.7
-0.5
-0.3
-0.1
0.1
0.3
0.5
0.7
.9
0


PPROXIMATE DENSITY FUNCTIONS
N RO1 RO2 SYMBOL
10 .50 .50 ⊡
20 .50 .50 ⊖
30 .50 .50 △
40 .50 .50 +
50 .50 .50 ×
F (X)
0.0
0.8
1.6
2.4
3.2
4
X
.9
-0.7
-0.5
-0.3
-0.1
0.1
0.3
0.5
0.7
0


APPROXIMATE DENSITY FUNCTIONS
N R01 R02 SYMBOL
30 .333 .750 ⊠
30 .417 .600 ⊖
30 .500 .500 △
30 .583 .429 +
30 .667 .375 ×
F (X)
X
0.0
0.8
1.6
2.4
3.2
4
-.9
-0.7
-0.5
-0.3
-0.1
0.1
0.3
0.5
0.7
0

3.2 The Calculation and Renormalization of Some Approximate Densities with Fitted Means

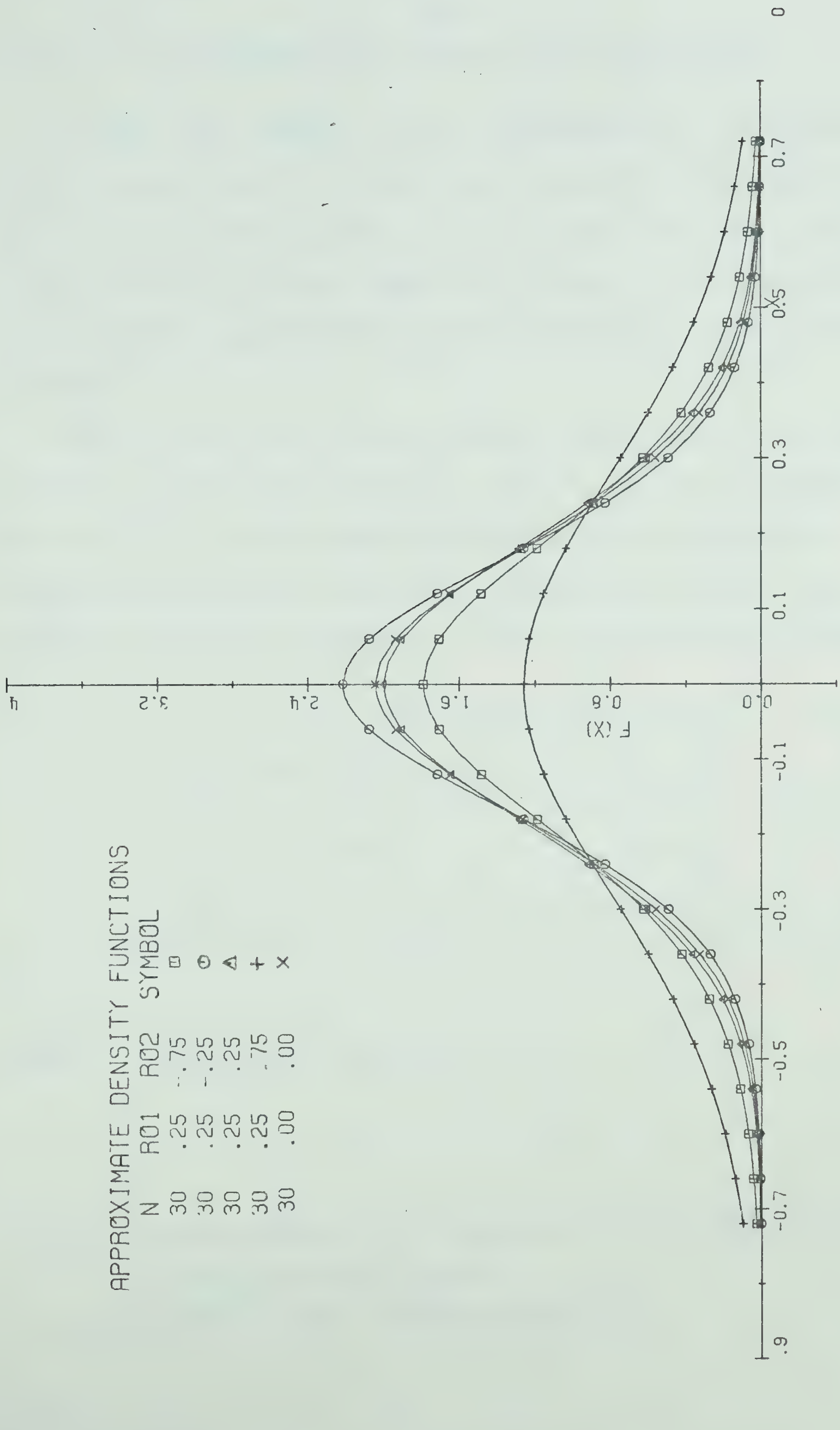
Using equations (2.5.25) to (2.5.28), once again noting the symmetry, one hundred ordinates were calculated and renormalized for b_1^* varying from -3 to +3.

Table 2 contains renormalizing constants corresponding to those in Table 1. The same trend for the constant to approach 1 with increasing n is apparent.

The same pattern in the behaviour of $f^*(b_1^*)$ was observed as in that of $f(b_1)$. Only one set of densities has been plotted. $\rho_1\rho_2$ has been held fixed, and ρ_1 allowed to increase from 1/3 to 2/3 in steps of 1/12. As would be expected from equation (2.6.16) for the variance of b_1^*, the variance decreases monotonically.

n	ρ_1	ρ_2	RC
10	0.50	0.50	1.64
20	0.50	0.50	1.23
30	0.50	0.50	1.16
40	0.50	0.50	1.13
50	0.50	0.50	1.11
30	-0.75	-0.75	1.04
30	-0.75	0.75	1.10
30	0.75	-0.75	1.15
30	0.75	0.75	1.42
30	0.0	0.0	1.04

Table 2 : Renormalization Constants : Fitted Means Case


APPROXIMATE DENSITY FUNCTIONS
N R01 R02 SYMBOL
30 .25 -.75 □
30 .25 -.25 ⊖
30 .25 .25 △
30 .25 .75 +
30 .00 .00 ×
F (X)
0.0
0.8
1.6
2.4
3.2
4
X
-.9
-0.7
-0.5
-0.3
-0.1
0.1
0.3
0.5
0.7

3.3 The Results of a Monte Carlo Simulation

To test the accuracy of the approximation, it was decided that, rather than run a large and expensive simulation to estimate the entire density function, the accuracy of the 0.025 percentage points should be tested. It is these values that would be used in the standard tests and confidence intervals.

With $\rho_1 = \rho_2 = 0.5$, four thousand values of b_1^* were generated for each of the cases, n = 10, 20, and 30. The proportion of b_1^*'s, in each case, greater than the approximate .025 percentage point was computed. The results, together with the statistic used for testing p = 0.025, are presented below.

ρ_1	ρ_2	n	p *	z †
0.5	0.5	10	0.03125	2.53
0.5	0.5	20	0.02475	-0.10
0.5	0.5	30	0.02250	-1.01

Table 3 : Results of the Simulation

* p = the proportion of b_1^*'s greater than the approximate .025 percentage point.

† $z = (p-.025)/[(.025)(.975)/(4000)]^{1/2}$.

Thus, it appears that, for the purpose of calculating percentage points commonly used in testing hypotheses and in constructing confidence intervals, the approximation is reasonably good for samples of size 30 (and perhaps smaller). Larger simulations considering different values of n, ρ_1, and ρ_2 and different percentage points are needed to make this statement more precise.

3.4 Conclusions

The approximate density of the regression coefficient has been found to be symmetric about 0, and approximately bell-shaped. The variance was found to depend on ρ_1 and ρ_2, but still to be of order $(1/n)$. Hence, the unbiasedness and consistency of the least squares estimator is confirmed. Equations (2.6.15) and (2.6.16), however, demonstrate that the level of the classical tests on β_1, and the width of the classical confidence intervals are likely to be inaccurate.

[1] Aitken, A. C. Determinants and Matrices. Edinburgh: Oliver and Boyd Ltd., 1959.

[2] Anderson, T. W. The Statistical Analysis of Time Series. John Wiley and Sons, Inc., 1971.

[3] Bielenstein, U. M. "The Approximate Distribution of the Correlation between Two Stationary, Linear, Markov Series with Fitted Means." M. Sc. thesis (unpublished), University of Alberta, 1963.

[4] Daniels, H. E. "Saddlepoint Approximations in Statistics." Annals of Mathematical Statistics, 1954, 25, 631-650.

[5] Daniels, H. E. "The Approximate Distribution of Serial Correlation Coefficients." Biometrika, 1956, 43, 169-185.

[6] DeBruijn, N. G. Asymptotic Methods in Analysis. North-Holland Publishing Co., 1961.

[7] Fisher, R. A. "Frequency-distribution of the Correlation Coefficient in Samples from an Indefinitely Large Population." Biometrika, 10, 507, 1915.

[8] Kendall, M. G., and Stuart, S. K. *The Advanced Theory of Statistics, Vol. 1*. Gharles Griffin and Co. Ltd., 1958.

[9] Malinvaud, E. *Statistical Methods in Econometrics*. North Holland Publishing Co., 1966.

[10] McGregor, J. R. "The Approximate Distribution of Certain Ratios in Time Series Analysis." Ph. D. thesis (unpublished), University of Cambridge, 1959.

[11] McGregor, J. R. "The Approximate Distribution of the Correlation between Two Stationary, Linear, Markov Series." *Biometrika*, 1962, *49*, 379-388.

[12] Muir, T. *A Treatise on the Theory of Determinants*. New York: Dover, 1960.

[13] Watson, G. S. *Serial Correlation in Regression Analysis*. Institute of Statistics, North Carolina State College. Mimeo. Series, No. *49*, 1951.

APPENDIX

DETAILS OF THE COMPUTATIONS

Programs were written in Fortran, for use in the Fortran G compiler. Double precision variables were found to be needed in the calculation of the approximate densities for $|\rho_1|$ or $|\rho_2|$ less than 0.1. The calculations are now accurate for $|\rho_1|$ and $|\rho_2|$ as small as 0.01. Graphs were plotted using "Auto-plotter".

The "gamma" function, needed in the calculation of the approximate densities when $\rho_1 = \rho_2 = 0$, was computed using the subroutine, "GMMMA", in the "Scientific Subroutine Package". Normal random deviates were generated using the subroutine, "CS:003A", of the Computing Centre of the University of Alberta.

Annotated programs, in the order in which their results appear in the text, are reproduced on the following pages.

```
C     UP TO FIVE APPROXIMATE DENSITY FUNCTIONS ARE
C     CALCULATED (FOR KNOWN MEANS CASE) AND
C     RENORMALIZED. INTEGRATION IS DONE USING
C     SIMPSON'S RULE. THE RENORMALIZING CONSTANTS
C     ARE PRINTED OUT, FOLLOWED BY A MATRIX OF
C     6 COLUMNS, THE FIRST OF WHICH CONTAINS
C     ABSCISSAE.THE REMAINING COLUMNS CONTAIN
C     CORRESPONDING ORDINATES OF RENORMALIZED
C     DENSITY FUNCTIONS.
      DIMENSION X(100),Y(100,5),N(5),RO1(5),RO2(5),RENOR(5)
      EXTERNAL V,Z,XI
      DIMENSION YT(100)
C     GMMMA IS THE SSP SUBPROGRAM WHICH EVALUATES
C     THE GAMMA FUNCTION.
      EXTERNAL GMMMA
C     MMAX IS THE NUMBER OF DENSITY FUNCTIONS EVALUATED.
      MMAX=1
      DO 9 I=1,MMAX
C     N IS THE SAMPLE SIZE.
C     RO1 IS THE AUTOCORRELATION OF THE X'S.
C     RO2 IS THE AUTOCORRELATION OF THE Y'S.
    9 READ 109,N(I),RO1(I),RO2(I)
  109 FORMAT (I5,2F5.2)
C     K EQUALS HALF THE NUMBER OF ABSCISSAE.
      K=50
C     THE RANGE OF THE ABSCISSAE IS (-OMAX,+OMAX).
      OMAX=3
      DO 1 I=0,K
      J=50+I
      L=50-I
      AI=I
      AK=K
C     THE X'S ARE DEFINED TO BE EVENLY SPACED OVER (-OMAX,+OMAX).
      X(J)=AI*OMAX/AK
      X(L)=-X(J)
      DO 1 M=1,MMAX
C     ORDINATES ARE CALCULATED USING THE EVEN SYMMETRY.
      Y(J,M)=F(X(J),RO1(M),RO2(M),V,Z,XI,N(M))
    1 Y(L,M)=Y(J,M)
      DO 7 M=1,MMAX
      DO 13 L=1,100
   13 YT(L)=Y(L,M)
C     RENOR IS THE AREA UNDER THE DENSITY FUNCTION,F.
    7 RENOR(M)=SIMP(OMAX,X,YT,K)
C     THE RENORMALIZATION CONSTANTS ARE PRINTED OUT.
      WRITE (5,101) RENOR
      DO 2 I=0,K
      DO 2 M=1,MMAX
      J=50+I
      L=50-I
C     THE ORDINATES ARE RENORMALIZED BY
C     DIVIDING BY RENOR.
      Y(J,M)=Y(J,M)/RENOR(M)
```

```
    2  Y(L,M)=Y(J,M)
       DO 3 I=1,100
C      THE MATRIX DESCRIBED ABOVE IS PRINTED OUT.
     3 WRITE (6,101) X(I),Y(I,1),Y(I,2),Y(I,3),Y(I,4),Y(I,5)
  101  FORMAT (6F13.6)
       STOP
       END
       FUNCTION XI(OMEGA,RO1,RO2)
C      SEE EQUATION 2.5.23.
       XI=OMEGA**2+SQRT((OMEGA**2+((1-RO1)/(1-RO2))**2)
      1*(OMEGA**2+((1+RO1)/(1+RO2))**2))
       RETURN
       END
       FUNCTION V(OMEGA,RO1,RO2,XI)
C      SEE EQUATION 2.5.22.
       XIV=XI(OMEGA,RO1,RO2)
       V=(XIV*(1.-RO2)**2+(1.-RO1)**2)*(XIV
      1*(1.+RO2)**2+(1.+RO1)**2)/8./RO1/RO2/(
      2XIV-OMEGA**2)
       RETURN
       END
       FUNCTION Z(OMEGA,RO1,RO2,XI,V)
C      SEE EQUATION 2.5.21.
       Y=V(OMEGA,RO1,RO2,XI)
       IF(RO1*RO2) 3,3,5
    3  Z=1.+Y+(Y**2+2.*Y)**.5
       RETURN
    5  Z=1.+Y-(Y**2+2.*Y)**.5
       RETURN
       END
       FUNCTION F(OMEGA,RO1,RO2,V,Z,XI,N)
C      SEE EQUATION 2.5.20.
       IF(RO1.EQ.0.0.OR.RO2.EQ.0.0) GO TO 5
       XIF=XI(OMEGA,RO1,RO2)
       VF=V(OMEGA,RO1,RO2,XI)
       ZF=Z(OMEGA,RO1,RO2,XI,V)
       FN=N
       F=8.*FN*(FN-2.)*SQRT((ABS(RO1*RO2))*(1-RO1**2)*(1-RO2**2)
      1)/(3.1415926*(FN-3.))**0.5/(FN-4.)/(1.-RO2**2)
       F=F*((VF**2+2.*VF)*(OMEGA**2+((1-RO1)/(1-RO2))**2)
      1*(OMEGA**2+((1+RO1)/(1.+RO2))**2))**0.25
       F=F*(1-ZF**2)*(1-ZF)**2/(1-RO1*RO2*ZF)
       F=F/(XIF*(1.-RO2**2)+(1-RO1**2))/(XIF*(1+RO2**2)
      1+(1+RO1**2))/(XIF**2*(1-RO2**2)**2+(1-RO1**2)**2
      2+2.*XIF*((RO1-RO2)**2+(1.-RO1*RO2)**2))**0.5
       F=F*(ZF/RO1/RO2)**((FN-3)/2)
       RETURN
    5  IF(RO1.EQ.0.0) GO TO 4
       RETURN
    4  IF(RO2.EQ.0.0) GO TO 6
       RETURN
    6  TN=N+1
       TN=TN/2.
       BN=N
```

```
      BN=BN/2.
      CALL GMMMA(TN,GT,0)
      CALL GMMMA(BN,GB,0)
      F=(GT/GB/3.1415926**.5)/(1+OMEGA**2)**TN
      RETURN
      END
      FUNCTION SIMP(OMAX,X,Y,K)
C     SIMPSON'S RULE NUMERICAL INTEGRATION ROUTINE.
      DIMENSION X(100),Y(100)
      SK=K
      H=OMAX/SK
      SUMEND=0.
      SUMMID=0.
      K1=2*K-1
      DO 8 J=0,K1,2
      L=J
      M=J+1
      SUMEND=SUMEND+Y(L)
    8 SUMMID=SUMMID+Y(M)
      M=2*K
      SIMP=(2.*SUMEND+4.*SUMMID-Y(0)+Y(M))*H/3.
      RETURN
      END
```

```
C     UP TO FIVE APPROXIMATE DENSITY FUNCTIONS ARE
C     CALCULATED (FOR FITTED MEANS CASE) AND
C     RENORMALIZED. INTEGRATION IS DONE USING
C     SIMPSON'S RULE. THE RENORMALIZING CONSTANTS
C     ARE PRINTED OUT, FOLLOWED BY A MATRIX OF
C     6 COLUMNS, THE FIRST OF WHICH CONTAINS
C     ABSCISSAE.THE REMAINING COLUMNS CONTAIN
C     CORRESPONDING ORDINATES OF RENORMALIZED
C     DENSITY FUNCTIONS.
      DIMENSION X(100),Y(100,5),N(5),RO1(5),RO2(5),RENOR(5)
      EXTERNAL V,Z,XI
      DIMENSION YT(100)
C     GMMMA IS THE SSP SUBPROGRAM WHICH EVALUATES
C     THE GAMMA FUNCTION.
      EXTERNAL GMMMA
C     MMAX IS THE NUMBER OF DENSITY FUNCTIONS EVALUATED.
      MMAX=5
      DO 9 I=1,MMAX
C     N IS THE SAMPLE SIZE.
C     RO1 IS THE AUTOCORRELATION OF THE X'S.
C     RO2 IS THE AUTOCORRELATION OF THE Y'S.
    9 READ 109,N(I),RO1(I),RO2(I)
  109 FORMAT (I5,2F5.2)
C     K EQUALS HALF THE NUMBER OF ABSCISSAE.
      K=50
C     THE RANGE OF THE ABSCISSAE IS (-OMAX,+OMAX).
      OMAX=3
      DO 1 I=0,K
      J=50+I
      L=50-I
      AI=I
      AK=K
C     THE X'S ARE DEFINED TO BE EVENLY SPACED OVER (-OMAX,+OMAX).
      X(J)=AI*OMAX/AK
      X(L)=-X(J)
      DO 1 M=1,MMAX
C     ORDINATES ARE CALCULATED USING THE EVEN SYMMETRY.
      Y(J,M)=F(X(J),RO1(M),RO2(M),V,Z,XI,N(M))
    1 Y(L,M)=Y(J,M)
      DO 7 M=1,MMAX
      DO 13 L=1,100
   13 YT(L)=Y(L,M)
C     RENOR IS THE AREA UNDER THE DENSITY FUNCTION,F.
    7 RENOR(M)=SIMP(OMAX,X,YT,K)
C     THE RENORMALIZATION CONSTANTS ARE PRINTED OUT.
      WRITE (5,101) RENOR
      DO 2 I=0,K
      DO 2 M=1,MMAX
      J=50+I
      L=50-I
C     THE ORDINATES ARE RENORMALIZED BY
C     DIVIDING BY RENOR.
      Y(J,M)=Y(J,M)/RENOR(M)
```

```
    2  Y(L,M)=Y(J,M)
       DO 3 I=38,62
C      THE MATRIX DESCRIBED ABOVE IS PRINTED OUT.
    3  WRITE (6,101) X(I),Y(I,1),Y(I,2),Y(I,3),Y(I,4),Y(I,5)
  101  FORMAT (6F13.6)
       STOP
       END
       FUNCTION XI(OMEGA,RO1,RO2)
C      SEE EQUATION 2.5.23.
       XI=OMEGA**2+SQRT((OMEGA**2+((1-RO1)/(1-RO2))**2)
      1*(OMEGA**2+((1+RO1)/(1+RO2))**2))
       RETURN
       END
       FUNCTION V(OMEGA,RO1,RO2,XI)
C      SEE EQUATION 2.5.22.
       XIV=XI(OMEGA,RO1,RO2)
       V=(XIV*(1.-RO2)**2+(1.-RO1)**2)*(XIV
      1*(1.+RO2)**2+(1.+RO1)**2)/8./RO1/RO2/(
      2XIV-OMEGA**2)
       RETURN
       END
       FUNCTION Z(OMEGA,RO1,RO2,XI,V)
C      SEE EQUATION 2.5.21.
       Y=V(OMEGA,RO1,RO2,XI)
       IF(RO1*RO2) 3,3,5
    3  Z=1.+Y+(Y**2+2.*Y)**.5
       RETURN
    5  Z=1.+Y-(Y**2+2.*Y)**.5
       RETURN
       END
       FUNCTION F(OMEGA,RO1,RO2,V,Z,XI,N)
C      SEE EQUATION 2.5.20.
       IF(RO1.EQ.0.0.OR.RO2.EQ.0.0) GO TO 5
       XIF=XI(OMEGA,RO1,RO2)
       VF=V(OMEGA,RO1,RO2,XI)
       ZF=Z(OMEGA,RO1,RO2,XI,V)
       FN=N-1
       F=8.*FN*(FN-2.)*SQRT((ABS(RO1*RO2))*(1-RO1**2)*(1-RO2**2)
      1)/(3.1415926*(FN-3.))**0.5/(FN-4.)/(1.-RO2**2)
       F=F*((VF**2+2.*VF)*(OMEGA**2+((1-RO1)/(1-RO2))**2)
      1*(OMEGA**2+((1+RO1)/(1.+RO2))**2))**0.25
       F=F*(1-ZF**2)*(1-ZF)**2/(1-RO1*RO2*ZF)
       F=F/(XIF*(1.-RO2**2)+(1-RO1**2))/(XIF*(1+RO2**2)
      1+(1+RO1**2))/(XIF**2*(1-RO2**2)**2+(1-RO1**2)**2
      2+2.*XIF*((RO1-RO2)**2+(1.-RO1*RO2)**2))**0.5
       F=F*(ZF/RO1/RO2)**((FN-3)/2)
       F=F*(1+ZF)/(1-RO1)/(1-RO2)
       F=F*(XIF*(1-RO2)**2+(1-RO1)**2)
       F=F/(XIF*(1+RO2**2)+(1+RO1**2))
       RETURN
    5  IF(RO1.EQ.0.0) GO TO 4
       RETURN
    4  IF(RO2.EQ.0.0) GO TO 6
       RETURN
```

```
    6 TN=N
      TN=TN/2.
      BN=N-1
      BN=BN/2.
      CALL GMMMA(TN,GT,0)
      CALL GMMMA(BN,GB,0)
      F=(GT/GB/3.1415926**.5)/(1+OMEGA**2)**TN
      RETURN
      END
      FUNCTION SIMP(OMAX,X,Y,K)
C     SIMPSON'S RULE NUMERICAL INTEGRATION ROUTINE.
      DIMENSION X(100),Y(100)
      SK=K
      H=OMAX/SK
      SUMEND=0.
      SUMMID=0.
      K1=2*K-1
      DO 8 J=0,K1,2
      L=J
      M=J+1
      SUMEND=SUMEND+Y(L)
    8 SUMMID=SUMMID+Y(M)
      M=2*K
      SIMP=(2.*SUMEND+4.*SUMMID-Y(0)+Y(M))*H/3.
      RETURN
      END
```

```
      DIMENSION B(4000)
      EXTERNAL CS003D
      READ(5,101) IXT,NS
C     IXT IS AN ARBITRARY INTEGER REQUIRED BY CS003A
C     TO START GENERATING NORMAL DEVIATES.
C     NS IS THE NUMBER OF REGRESSION COEFFICIENTS CALCULATED.
      CALL CS003A(IXT)
      READ(5,107) N,RO1,RO2,BC
C     N IS THE SAMPLE SIZE USED TO CALCULATE EACH REGRESSION COEFFICIENT.
C     RO1 AND RO2 ARE THE AUTOCORRELATIONS.
C     BC IS THE .025 PERCENTAGE POINT OF THE APPROXIMATE DENSITY FUNCTION
  107 FORMAT(I2,2F5.2,F10.6)
  101 FORMAT (I6,I5)
      DO 1 K=1,NS
    1 B(K)=BMC(N,RO1,RO2)
C     B IS A VECTOR OF LENGTH NS CONSISTING OF
C     THE GENERATED REGRESSION COEFFICIENTS.
      BR=0.
      DO 8 I=1,NS
      IF(B(I).GT.BC) BR=BR+1.
C     BR IS THE NUMBER OF REGRESSION COEFFICIENTS GREATER THAN BC.
    8 CONTINUE
      FNS=NS
      BPR=BR/FNS
C     BPR IS THE PROPORTION OF REGRESSION COEFFICIENTS GREATER THAN BC.
      WRITE(6,108) BPR
  108 FORMAT(F10.6)
      STOP
      END
      FUNCTION BMC(N,RO1,RO2)
C     THE FUNCTION BMC GENERATES REGRESSION COEFFICIENTS
C     FOR SAMPLES OF SIZE UP TO 50.
      DIMENSION X(50),Y(50)
      DO 2 I=1,N
      CALL CS003D(0.,1.,U,V,KN)
      IF(I.EQ.1) GO TO 3
      J=I-1
      X(I)=RO1*X(J)+U
      Y(I)=RO2*Y(J)+V
      XY=XY+X(I)*Y(I)
      XX=XX+X(I)
      YY=YY+Y(I)
      XXX=XXX+X(I)**2
      GO TO 2
    3 X(I)=U
      Y(I)=V
      XY=X(I)*Y(I)
      XX=X(I)
      YY=Y(I)
      XXX=X(I)**2
    2 CONTINUE
      BMC=(N*XY-XX*YY)/(N*XXX-XX**2)
  102 FORMAT (F13.6)
      RETURN
      END
```